Bachelor of Science
(B.Sc. - PCM)

Differential Calculus
(DBSPCO103T24)

Self-Learning Material
(SEM 1)

Jaipur National University
Centre for Distance and Online Education

Established by Government of Rajasthan
Approved by UGC under Sec 2(f) of UGC ACT 1956
&
NAAC A+ Accredited

PREFACE

Differential calculus is a foundational branch of mathematics with far-reaching implications in various fields, including physics, engineering, economics, and computer science. It serves as a fundamental tool for understanding rates of change, optimization, and the behaviour of functions.

This book is designed to be a comprehensive resource for students, educators, and anyone interested in mastering the principles and applications of differential calculus. This book aims to provide a clear and structured approach to learning differential calculus, starting from the basics and gradually progressing to advanced topics. Whether you're a novice seeking an introduction to calculus or a seasoned mathematician looking to deepen your understanding, this book offers something for everyone.

TABLE OF CONTENTS

CHAPTER - 1

Numbers and functions

Learning Objectives:

- Differentiate between types of numbers (natural, whole, integers, rational, irrational, real and complex).
- Identify key features of functions (domain, range) from different representations.
- Recognize common function types (linear, quadratic, exponential, logarithmic, trigonometric)

Structure:

We use numbers in our day to day life. They are often called numerals. Without numbers, we cannot do counting of things, date, time, money, etc. Sometimes these numbers are used for measurement and sometimes they are used for labelling. The properties of numbers make them capable of performing arithmetic operations on them. These numbers are expressed in numeric forms and also in words. For example, 2 is written as two in words, 25 is written as twenty-five in words, etc. Students can practice writing the numbers from 1 to 100 in words to learn more.

There are different types of numbers in Maths, which we learn. They are natural and whole numbers, odd and even numbers, rational and irrational numbers, etc. We will discuss all the types here in this article. Apart from these, the numbers are used in various applications such as forming number series, maths tables, etc.

1.1 Numbers Introduction

A number is an arithmetic value used for representing the quantity and used in making calculations. A written symbol like "3" which represents a number is known as numerals. A number system is a writing system for denoting numbers using digits or symbols in a logical manner. The numeral system:

- Represents a useful set of numbers
- Reflects the arithmetic and algebraic structure of a number
- Provides standard representation

We use the digits from 0 to 9 to form all other numbers.

0	1	2	3	4	5	6	7	8	9

With the help of these digits, we can create infinite numbers.
For example, 12, 3456, 1298, etc.

Counting Numbers:

We use numbers to count different things or objects such as 1, 2, 3, 4, etc. Humans have been using numbers to count things from the past thousands of years. For example, there are 7 cows in the field. The counting numbers start from 1 and it goes till infinity.

The Number Zero:

The position of the number "Zero (0)" plays an important role in Mathematics and it is used as a placeholder in the place value number system. The number 0, acts as an additive identity for the real numbers, and other algebraic structures. We use the number "0" to show nothing. For example, there were 3 apples, but now there are none. To represent nothing, we can use zero.

Types of Numbers

The numbers can be classified into sets known as the number system. The different types of numbers in maths are:

- **Natural Numbers:** Natural numbers are known as counting numbers that contain the positive integers from 1 to infinity. The set of natural numbers is denoted as "N" and it includes N = {1, 2, 3, 4, 5, ……….}

- **Whole Numbers:** Whole numbers are known as non-negative integers and it does not include any fractional or decimal part. It is denoted as “W” and the set of whole numbers includes W = {0, 1, 2, 3, 4, 5, ……….}
- **Integers:** Integers are the set of all whole numbers but it includes a negative set of natural numbers also. “Z” represents integers and the set of integers are Z = { -3, -2, -1, 0, 1, 2, 3}
- **Rational Numbers:** Any number that can be written as a ratio of one number over another number is written as rational numbers. This means that any number that can be written in the form of p/q. The symbol “Q” represents the rational number.
- **Irrational Numbers:** The number that cannot be expressed as the ratio of one over another is known as irrational numbers and it is represented by the symbol “P”
- **Real Numbers:** All the positive and negative integers, fractional and decimal numbers without imaginary numbers are called real numbers. It is represented by the symbol “R”.
- **Complex Numbers:** The number that can be written in the form of $a + bi$, where “a and b” are the real number and “i” is an imaginary number, are known as complex numbers “C”.
- **Imaginary Numbers:** The imaginary numbers are the complex numbers that can be written in the form of the product of a real number and the imaginary unit “i”.

Apart from the above, there exist other numbers namely even and odd numbers, prime numbers and composite numbers. These can be defined as given below:

Even Numbers: The numbers which are exactly divisible by 2, are called even numbers. These can be positive or negative integers such as -42, -36, -12, 2, 4, 8 and so on.

Odd Numbers: The numbers which are not exactly divisible by 2, are called odd numbers. These can be both positive and negative integers such as -3, -15, 7, 9, 17, 25 and so on.

Prime Numbers: Prime numbers are the numbers that have two factors only i.e., 1 and the number itself. In other words, the number which is divided by 1 and the number itself is called prime numbers. For example 2, 3, 5, 7, 11 etc.

Composite Numbers: A composite number is a number that has more than two factors. For example, 4 is a composite number, as the number 4 is divisible by 1, 2, and 4. Other examples of composite numbers are 6, 8, 9, 10, and so on.

Note: The number “1” is neither prime nor composite.

Numbers Chart: The number chart represents Real and Complex numbers (Fig. 1.1.1)

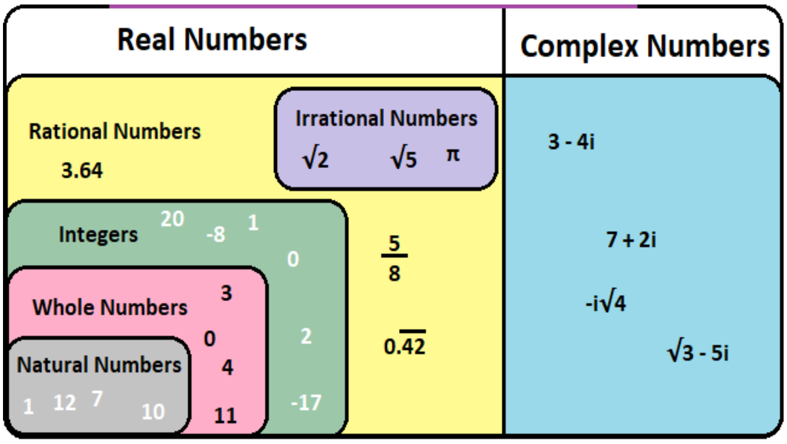


Figure 1.1.1: Number Chart

Number Series

In mathematics, the number series consists of a series of numbers in which the next term is obtained by adding or subtracting the constant term to the previous term. For example, consider the series 1, 3, 5, 7, 9, … In this series, the next term is obtained by adding the constant term "2" to the previous term. There are different types of number series namely,

- Perfect Square series
- Two-stage type series
- The odd man out series
- Perfect cube series
- Geometric series
- Mixed series

Special Numbers

Cardinal Numbers: Cardinal number defines how many of something are there in a list, such as one, five, ten, etc.

Ordinal Numbers: Ordinal numbers explain the position of something in a list, such as first, second, third, fourth, and so on.

Nominal Numbers: Nominal number is used only as a name. It does not denote an actual value or the position of something.

Pi (π): Pi is a special number, which is approximately equal to 3.14159. Pi (π) is defined as the ratio of the circumference of the circle divided by the diameter of the circle.

(i.e.,) Circumference/ Diameter = π = 3.14159.

Euler's Number: Euler's number is one of the important numbers in Maths, and it is approximately equal to 2.7182818. It is an irrational number and it is the base of the natural logarithm.

Golden Ratio: A golden ratio is a special number and it is approximately equal to 1.618. It is an irrational number and the digits do not follow any pattern.

Properties of Numbers

The properties of numbers are basically stated for real numbers. The common properties are:

Commutative Property: If a and b are two real numbers, then according to commutative property;

a+b = b+a

a.b = b.a

Example: 2+3 = 3+2

and 2 × 3 = 3 × 2

Associative Property: If a, b and c are three real numbers, then according to associative property;

(a+b)+c = a+(b+c)

(a.b).c = a.(b.c)

Example: (1+2) +3 = 1+ (2+3)

(1.2).3 = 1.(2.3)

Distributive Property: If a, b and c are three real numbers, then according to distributive property;

a×(b + c) = a×b + a×c

Example: 2 × (3 + 4) = 2×3 + 2×4

2 × 7 = 6 + 8

14 = 14

Closure Property: If a number is added to another number, then the outcome will be a number only, such as;

a+b = c; where a, b and c are three real numbers.

Example: 1+2 = 3

Identity Property: If we add zero to a number or multiply by 1, the number will remain unchanged.

a+0=a

a.1 =a

Example: 5+0 = 5 and 5 x 1 = 5

Additive Inverse: If a number is added to its own negative number, then the result is zero.

a+(-a) = 0

Example: 3+(-3) = 3-3 = 0

Multiplicative Inverse: If a number apart from 0, is multiplied to its own reciprocal then the result is 1.

a × (1/a) = 1

Example: 23 × (1/23) = 1

Zero Product Property: If a.b = 0, then; either a = 0 or b = 0.

Example: 7 × 0 = 0 or 0 × 6 = 6

Reflexive Property: This property reflects the number itself.

a = a

Example: 9 = 9

The properties which are explained above can vary based on the different types of numbers.

Solved Problems:

Example 1.1.1:

Prove the associative property of addition and multiplication.

Solution:

We know that the associative property of addition and multiplication are:

(a+b)+c = a+(b+c)

(a.b).c = a.(b.c)

Now, assume that a = 2, b = 4 and c = 5

Proving associative property of addition:

Now, substitute the values in the property

(2+4)+5 = 2+(4+5)

6+5 = 2+9

11 = 11

L.H.S = R.H.S

Hence, (a+b)+c = a+(b+c) is proved.

Proving associative property of multiplication:

(2.4).5 = 2.(4.5)

(8).5 = 2.(20)

40 = 40

L.H.S = R.H.S

Hence, (a.b).c = a.(b.c) is proved.

Example 1.1.2:

Solve the given algebraic expression 4.(3+2) using the distributive property

Solution:

We know that the distributive property is $a \times (b + c) = (a\times b) + (a\times c)$

Now, take a= 4, b= 3 and c= 2

Now, substituting values, we get

$4.(3+2) = (a\times b) + (a\times c)$

$= (4\times 3) + (4\times 2)$

= 12+8

= 20

Hence, 4.(3+2) is 20.

1.2 Functions

The relationship between the particle location and time, the relationship between a point on the x- and y-axes, and several other similar relationships are investigated in-depth in the name function.

Assume that a particle is travelling across space. The physical particle is taken to be a point. The particle's location fluctuates with time. From a mathematical perspective, the point is always located in the three-dimensional space R^3. Let time ranges from 0 to 1, then the particle's function or movement determines its position at any given time t, which ranges from 0 to 1. Stated differently, the functioning of the particle yields a point in R^3 for each $t \in [0, 1]$. Let f(t) represent the particle's location at time t.

Here's another simple illustration. We are aware that a straight line is described by the equation $2x - y = 0$. In this case, y takes on a value in accordance with whatever x assumes.

The way that x moves or functions determines how y moves. Let y be represented by f(x). Such situations are common in nature. It is evident that when studying natural events, one must take into account how one quantity varies in relation to another.

The relationship between the particle location and time, between a point on the x- and y-axes, and several other similar relationships are investigated in-depth in the name function. The definition of a function prior to Cantor is a rule that links one variable to another. A function is defined as a rule that associates a unique element in set B with each element in set A with the emergence of the idea of sets. Nevertheless, rule and association are not mathematical notions with precise definitions. Every word used in modern mathematics needs to be defined correctly. Thus, relations are used to provide a definition of function.

Let's say we intend to talk about an exam that a group of students wrote. We will consider this to be related. Let A represent the group of students who took the exam, and let B = {0, 1, 2, 3,... 100} represent the range of potential scores. A relation R is defined as follows:

A student *a* is related to a mark *b* if *a* got *b* marks in the test. When a student receives b marks on an exam, they are associated with a mark b. Each student achieved a grade. Put differently, for any $a \in A$, there exists an element $b \in B$ such that $(a, b) \in R$.

In any test, a student cannot receive two distinct scores. Put differently, there exists a single $b \in B$ such that $(a, b) \in R$ for each $a \in A$. Another way to put this is thus: $b = c$ if $(a, b), (a, c) \in R$. A significant class of relations known as functions is made up of relations with the two above mentioned characteristics.

Finally, let us have a precise specification of a function using relations.

Definition 1.2.1

Let A and B be two sets. A relation f from A to B, a subset of $A \times B$, is called a function from A to B if it satisfies the following

(i) For all $a \in A$, there is an element $b \in B$ such that $(a, b) \in f$.

(ii) If $(a, b) \in f$ and $(a, c) \in f$ then $b = c$.

Put another way, a function is a relation where every element in the domain is mapped to a single element in the range exclusively.

The terms "domain" and "co-domain" refer to A and B, respectively, of f. When (a, b) is in f, we express f(a) = b; this indicates that element ‘a’ is a pre-image of ‘b’ element ‘b’ is the image of ‘a’ and f(a) is the value of f at ‘a’. The function's range is the set

$\{b: (a, b) \in f \text{ for some } a \in A\}$.

It is said that the function is real-value if B is a subset of R. If the domains of two functions, f and g, are the same and f(a) = g(a) for every ‘a’ in the domain, then the two functions are considered equivalent. We write f: A → B if f is a function with domains A and B. (Read this as f being a function from A to B or as f being from A to B). The term "f maps A into B" is sometimes used. We say that f maps a to b or that f maps a onto b if f(a) = b, and so on.

The set of all co-domain elements with pre-images is referred to as a function range. It is obvious that a function's range is a portion of its co-domain. The domain of a relation R from a set A to a set B is defined as the set of all members of A having images, not as A, because the first condition further stipulates that each element in the domain must have an image. An element in the domain cannot have two or more photos, according to the second requirement. Although a relation does not always imply a function, we see that every function is a relation. Let $f = \{(a, 1), (b, 2), (c, 2), (d, 4)\}$ be a function. This function goes from {{a, b}, {c}, {d}} to {1, 2, 4}. Since *e* has no image, this function from {a, b, c, d, e} to {1, 2, 3, 4} cannot exist. Since the image of d is not in the co-domain and f is not a subset of {a, b, c, d}×{1, 2, 3, 5}, this function from {a, b, c, d} to {1, 2, 3, 5} is not defined.

1. Way of Representing Functions

(a) Tabular Representation of a Function

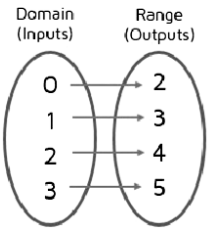
When the members of the domain are listed like $x_1, x_2, x_3, \ldots x_n$, we can use this tabular form. Here, the values of the arguments $x_1, x_2, x_3, \ldots x_n$ and the corresponding values of the function $y_1, y_2, y_3 \ldots y_n$ are written in a definite order.

x	x_1	x_2	$\ldots$	x_n
y	y_1	y_2	$\ldots$	y_n

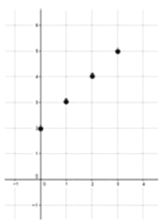
(b) Graphical Representation of a Function

A graph with x-axis for the domain and y-axis for the co-domain in the (x, y)-plane may be used to depict numerous functions when the domain and the co-domain are subsets of R.

Mapping Table Graph



x	y
0	2
1	3
2	4
3	5



$\{(0,2),(1,3),(2,4),(3,5)\}$

Figure: 1.2.1: Graphical Representation of a Function

(c) Analytical Representation of a Function

We say that the function y of x is represented or defined analytically if the function y = f(x) is such that f denotes an analytical phrase. Several instances of analytical phrases include

$$x^3 + 5, \ \frac{\sin x + \cos x}{x^2 + 1}, \ \log x + 5\sqrt{x}.$$

Illustrations of functions define are

$$\text{(i)} \ \ y = \frac{x - 1}{x + 1} \quad \text{(ii)} \ \ y = \sqrt{9 - x^2} \quad \text{(iii)} \ \ y = \sin x + \cos x \quad \text{(iv)} \ \ A = \pi r^2.$$

Now recall the domain of the function which are analytical in nature described as

(i) $y = 2x$, **(ii)** $y = x^2$, **(iii)** $y = +\sqrt{x}$, **(iv)** $y = -\sqrt{x}$

Piece- wise defined functions , considering the function f(x) defined as

$f : \mathbb{R} \to \mathbb{R}$

$$f(x) = \begin{cases} 0 & \text{if} \quad -\infty < x \le -2 \\ 2x & \text{if} \quad -2 < x \le 3 \\ x^2 & \text{if} \quad \ \ 3 < x \le \infty \end{cases}$$

We can create the function's graph if it is defined from R or a subset of R. For instance, we may display the points (x, x/2 + 1) for all ∈ [0, 4] if f: [0, 4] → R is defined by f(x) = x/2 + 1. After that, a straight line segment connecting (0, 1) and (4, 3) will appear. Refer to figure 1.2.

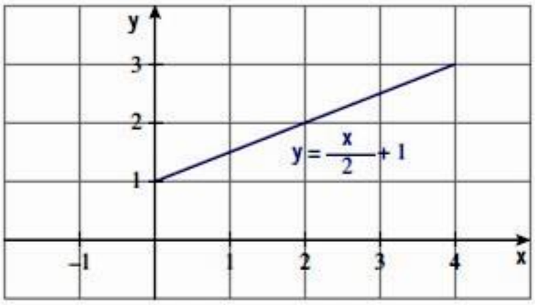


Figure 1.2.2: Straight line segment

Other example of function *f*(*x*) = x_2 + 4, *x* ≥ 0. The mapping will be specified by its graph

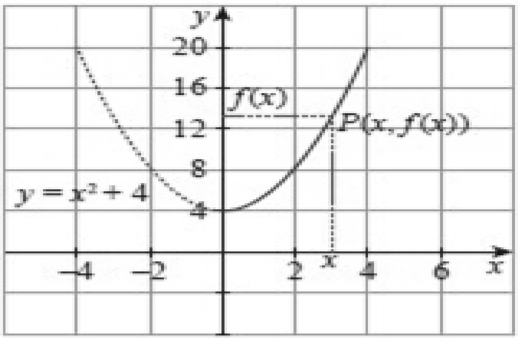


Figure: 1.2.3: Mapping *of* *f*(*x*) = x_2 + 4, *x* ≥ 0

A point in the domain is denoted by x. Now let us draw a line across point x that is vertical. At P, let it intersect the curve. f(x) is the place where the y-axis and the horizontal line drawn via P meet. In a similar manner, we may determine the pre-images of y by drawing horizontal lines across a point y in the co domain.

Is it possible to define a curve on a plane as a function that maps a subset of R to R? No, we are unable to. To determine this, there is a straightforward test.

Vertical Line Test

As we previously said, the y-coordinate of each point x in the domain is equal to f(x), where the vertical line across the point intersects the curve at that point. We will obtain many values for f(x) for a single x if the vertical line passing through a point x in the domain encounters the curve more than once. This isn't appropriate for a function. Furthermore, there won't be an image for x if the vertical line through a point x in the domain doesn't match the curve; this can't happen in a function either. Thus, we may state,

"if the vertical line through a point *x* in the domain meets the curve at more than one point or does not meet the curve, then the curve will not represent a function".

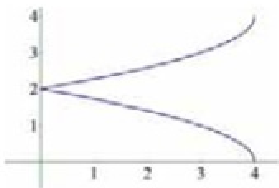


Figure 1.2.4 Arch depict

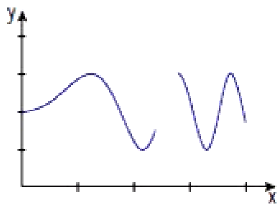


Figure 1.2.5 Arch depict

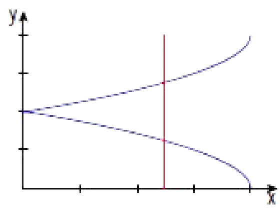


Figure 1.2.6 Arch depict

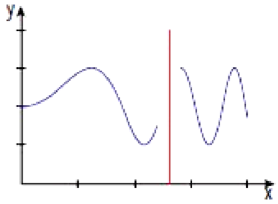


Figure 1.2.7 Arch depict

The Arch depict in Figure 1.2.4 does not symbolize a map from [0, 4] to R due to a vertical line meets the curve at greater than one point Figure 1.2.5. The graph shows in Figure 1.2.6 does not signify a function from [0, 4] to R due to a vertical line drawn through x = 2.5 in [0, 4] does not convene the graph Figure 1.2.7.

2. Various Elementary mapping

A few commonly used mapping have names attached to them. Let's enumerate a few of them.

(i) Let X be any non-empty set. The function $f : X \to X$ described by $f(x) = x$ for each $x \in X$ is identified the ***identity mapping*** on X (See Figure 1.2.4). It is represented by IX or I.

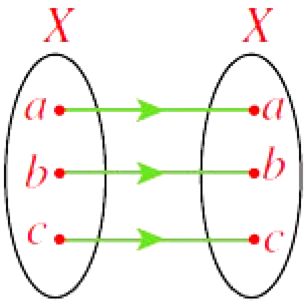


Figure 1.2.8 Mapping

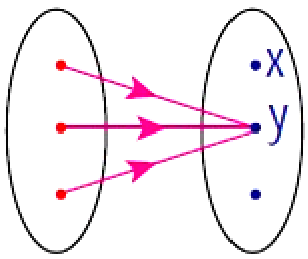


Figure 1.2.9 Mapping

(ii) Two sets, X and Y, are given. Assume that c is a fixed element in Y. A constant function is the mapping f: X → Y defined by f(x) = c for every x ∈ X (See Figure 1.2.9).

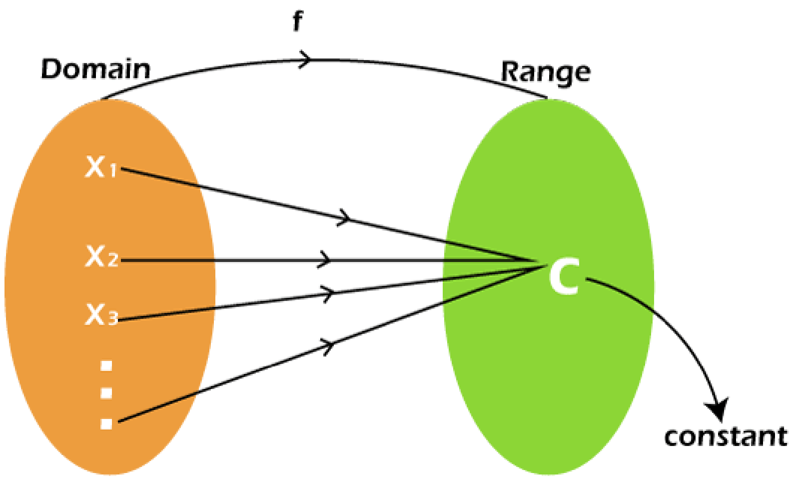


Figure 1.2.10 Constant Function

The value of a invariable mapping is same for all values of *x* throughout the domain.

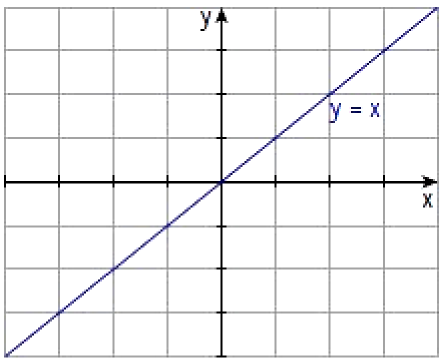


Figure 1.2.11: Identity function

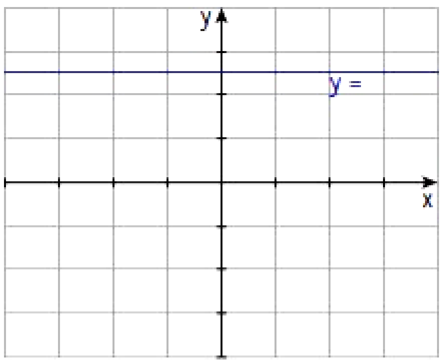


Figure 1.2.12: Identity function

The graphs of the identity function and a constant function are shown in Figures 1.2.11 and 1.2.12 if X and Y are both R. The zero function is the constant function defined by f(x) = 0 for every x, if X is any set. Thus, a specific instance of a constant function is the zero function.

Now for f(x) = |x|

$$|x| = \begin{cases} -x & \text{if} \quad x < 0 \\ 0 & \text{if} \quad x = 0 \\ x & \text{if} \quad x > 0 \end{cases}$$

$$|x| = \begin{cases} -x & \text{if} \quad x \le 0 \\ x & \text{if} \quad x > 0 \end{cases}$$

$$|x| = \begin{cases} -x & \text{if} \quad x < 0 \\ x & \text{if} \quad x \ge 0 \end{cases}$$

(iii) The map *f* : R → R defined by *f*(*x*) = |*x*|, where |*x*| is the modulus or absolute value of *x*, is called the **modulus mapping** or **absolute value function**. Figure 1.2.13 & Figure 1.2.14

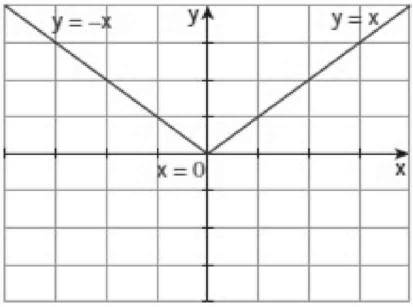


Figure 1.2.13: Modulus Mapping

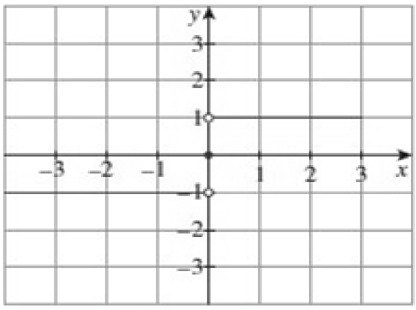


Figure 1.2.14: Modulus Mapping

3. Types of Functions

While there are many other forms of functions depending on the situation, we will focus on two fundamental categories: onto functions and one-to-one functions.

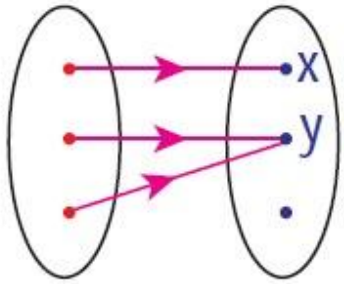


Figure 1.2.15 One-to-one functions

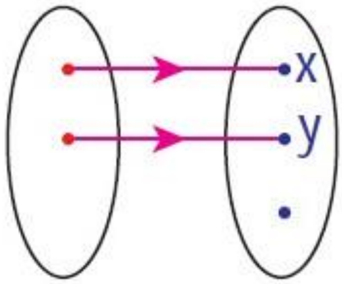


Figure 1.2.16 One-to-one functions

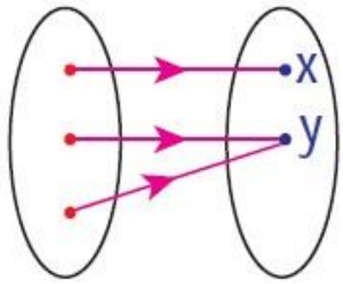


Figure 1.2.17 One-to-one functions

Let's examine the two straightforward functions shown in Figures 1.2.15 and 1.2.17. While this is not the case in Figure 1.2.16, two domain items, b and c, are mapped onto the same element, y, in the first function. One-to-one functions include those such as the second one.

Examining the two functions displayed in Figures 1.2.16 and 1.2.17 is our next task. The element z in the co-domain does not have a pre-image in Figure 1.2.16, but it does in Figure 1.2.17. Examples of onto functions are the functions shown in Figure 1.2.17. We now define onto and one-to-one functions.

Definition 1.2.2

A function f $A \rightarrow B$ is said to be one-to-one if $x, y \in A$, $x \neq y \Rightarrow f(x) \neq f(y)$ [or equivalently $f(x) = f(y) \Rightarrow x=y$]. A function $f: A \rightarrow B$ is said to be onto, if for each $b \in B$ there exists at least one element $a \in A$ such that $f(a) = b$. That is, the range of f is B.

An onto function is a surjective function, while an injective function is another term for a one-to-one function. If a function f: A → B is both onto and one-to-one, it is referred to as bijective.

In order to demonstrate that a function f: A → B is one-to-one, one just has to demonstrate one of the following:

If $x^{-1} y$, then $f(x)^{-1f}(y)$, or equally if $f(x) = f(y)$, then $x = y$.

Example 1.2.1: Check whether the following functions are one-to-one and onto.

i. $f: N \to N$ defined by $f(n) = n + 2$.

ii. $f: N \cup \{-1, 0\} \to N$ defined by $f(n) = n + 2$.

Solution:

i. If $f(n) = f(m)$, then $n + 2 = m + 2$ and hence m=n. Thus f is one –to-one. As 1 has no pre-image, this function is not onto.

ii. As above, this function is one-to-one, If m is in the co-domain, then $m - 2$ is in the domain and $f(m - 2) = (m - 2) + 2 = m$; thus m has a pre-image and hence this function is onto.

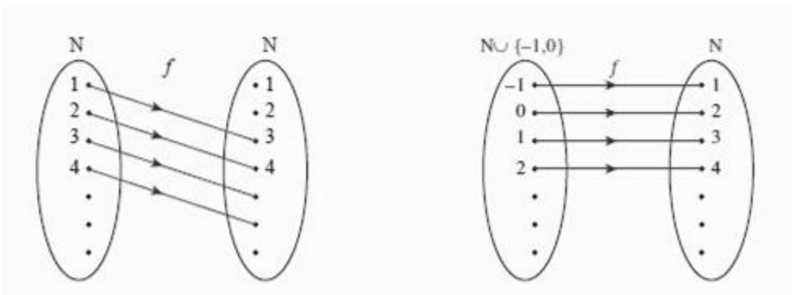


Figure 1.2.18 Mappings

1.3 Summary

Functions are fundamental concepts in mathematics and computer science. In mathematics, a function is a relation between a set of inputs and a set of possible outputs, with the property that each input is related to exactly one output.

1.4 Keywords

- Numbers
- Types of Numbers
- Number Series

- Functions
- Types of Functions
- Operations on Functions

1.5 Self – Assessment Questions

1. What is a mapping?
2. How is a mapping different from a function?
3. What are the components of a mapping?
4. Can mappings have multiple outputs for a single input?
5. What is the inverse of a mapping?
6. What is the domain and range of a mapping?
7. How do you represent a mapping graphically?
8. What is the difference between a one-to-one mapping and an onto mapping?
9. Give an example of a real-life mapping.
10.

1.6 Case Study

Geographic mapping plays a critical role in disaster response efforts, enabling responders to visualize affected areas, identify vulnerable populations, and coordinate rescue and relief operations efficiently. This case study illustrates the use of mapping technology in the aftermath of a natural disaster.

Objective: A powerful earthquake has struck a densely populated region, causing widespread destruction to infrastructure and displacing thousands of people. Emergency responders are mobilizing to provide aid and support to affected communities.

1.7 References

1. Dantzig, T. (2007). Number: The Language of Science (Revised edition). Dover Publications.
2. Stewart, J. (2015). *Calculus* (8th ed.). Brooks Cole.

CHAPTER - 2

Derivatives

Learning Objectives:

- Define what a derivative is and its significance in calculus.
- Understand and apply differentiation rules, including the power rule, product rule, quotient rule, and chain rule.
- Identify and analyze real-world applications of derivatives, such as optimization, curve sketching, and motion problems.

Structure:

2.1 Derivatives

A derivative in calculus is the rate of change of a quantity y with respect to another quantity x. It is also termed the differential coefficient of y with respect to x. Differentiation is the process of finding the derivative of a function.

Let us learn what exactly a derivative means in calculus and how to find it along with rules and examples.

Meaning of Derivatives in Calculus

The **derivation** of a function f(x) is usually represented by d/dx(f(x)) (or) df/dx (or) Df(x) (or) f'(x). Let us see what a derivative technically means. Consider a curve of function f(x) and let two points on it be (x, f(x)) and ((x + h), f(x + h)). Then the slope of the secant line through these points is given by [f(x + h) - f(x)]/(x + h - x) = [f(x + h) - f(x) / h. See the figure below and observe that when the distance between two points is closely equal to 0 (i.e., as h approaches 0), the second point overlaps the original point and the secant line becomes the tangent line. In calculus, the slope of the tangent line is referred to as the derivative of the function. i.e.,

- The derivation of the function, f '(x) = Slope of the tangent = $\lim_{h\to 0}$ [f(x + h) -f(x) / h.

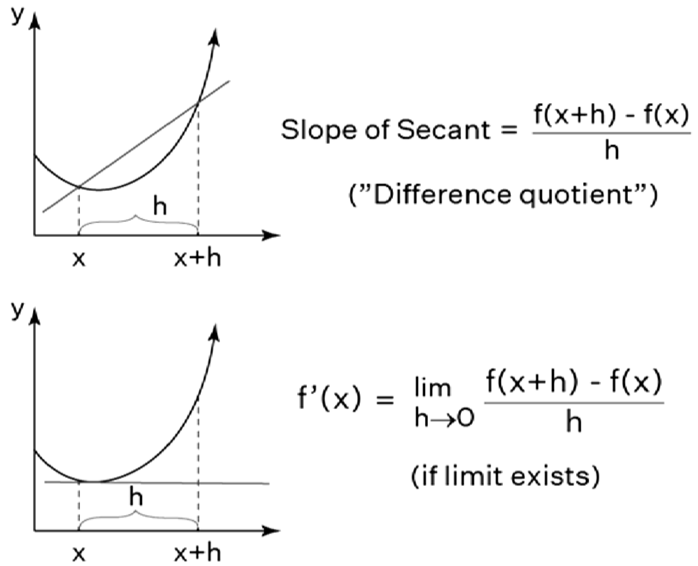


Figure 2.1.1 Derivative of function

This formula is popularly known as the "limit definition of the derivative" (or) "derivative by using the first principle".

Interpretation of Derivatives

The derivation of a mapping f(x) in math is denoted by f'(x) and can be contextually interpreted as follows:

- The derivation of a mapping at a point is the slope of the tangent drawn to that curve at that point.
- It also represents the instantaneous rate of change at a point on the map.
- The velocity of a particle is found by finding the derivative of the displacement function.
- The derivatives are used to optimize (maximize/minimize) a function.

Thus, whenever we see the phrases like "slope/gradient", "rate of change", "velocity (given the displacement)", "maximize/minimize" etc. Then it means that the concept of derivatives is involved.

Derivation of a Function Using the First Principle

The derivation of a function can be obtained by the limit definition of derivative which is f'(x) = $\lim_{h\to 0}$ [f(x + h) - f(x) / h. This process is known as the differentiation by the first principle. Let $f(x) = x^2$ and we will find its derivative using the above derivative formula. Here, $f(x + h) = (x + h)^2$ as we have $f(x) = x^2$. Then the derivative of f(x) is,

$f'(x) = \lim_{h\to 0} [(x + h)^2 - x^2] / h$

$= \lim_{h\to 0} [x^2 + 2xh + h^2 - x^2] / h$

$= \lim_{h\to 0} [2xh + h^2] / h$

$= \lim_{h\to 0} [h(2x + h)] / h$

$= \lim_{h\to 0} (2x + h)$

$= 2x + 0$

$= 2x$

Thus, the derivative of x^2 is 2x. But it may be difficult to use this limit definition to find the derivatives of complex functions. Thus, there are some derivative formulas (of course, which are derived from the above limit definition) that we can use readily in the process of differentiation.

Power Rule of Derivatives

By using the above example, the derivative of x^2 is 2x. Similarly, we can prove that the derivative of x^3 is $3x^2$, the derivative of x^4 is $4x^3$, and so on. Power rule generalizes this and it is stated as $d/dx\ (x^n) = n\ x^{n-1}$.

Derivatives of Log/Exponential Functions

- The derivative of ln x is, $d/dx\ (\ln x) = 1/x$
- The derivative of log x is, $d/dx\ (\log_a x) = 1/(x \ln a)$
- The derivative of e^x is, $d/dx\ (e^x) = e^x$
- The derivative of a^x is, $d/dx\ (a^x) = a^x \ln a$

Derivatives of Trigonometric Functions

$\frac{d}{dx}(\sin x) = \cos x$

$\frac{d}{dx}(\cos x) = -\sin x$

$\frac{d}{dx}(\tan x) = \sec^2 x$

$\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

$\frac{d}{dx}(\sec x) = \sec x \tan x$

$\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$

2.2 Fundamental Rules of Derivatives

The following are the fundamental rules of derivatives. Let us discuss them in detail.

Power Rule: By this rule, if $y = x^n$, then $dy/dx = n\, x^{n-1}$. Example: $d/dx\,(x^5) = 5x^4$.

Sum/Difference Rule: The derivative process can be distributed over addition/subtraction. i.e., $dy/dx\,[u \pm v]= du/dx \pm dv/dx$.

Product Rule: The product rule of derivatives states that if a function is a multiplication of two functions, then its derivation is the derivative of the second mapping multiplied by the first function added to the derivation of the first function multiplied by the second function. $dy/dx\,[u \times v] = u \cdot dv/dx + v \cdot du/dx$. If $y = x^5 e^x$, we have $y' = x^5 . e^x + e^x . 5x^4 = e^x (x^5 + 5x^4)$

Quotient Rule: The quotient rule of derivatives states that

$d/dx\,(u/v) = (v\cdot du/dx - u\cdot dv/dx)/v^2$

Constant multiple Rule: The constant multiple rule of derivatives states that $d/dx\,[c(f(x)] = c d/dx\, f(x)$. i.e., the constant which when multiplied by a function, comes out of the differentiation process. For example, $d/dx\,(5x^2) = 5\, d/dx\,(x^2) = 5(2x) = 10\,x$.

Constant Rule: The constant rule of derivatives state that the derivative of any constant is 0. If $y = k$, where k is a constant, then $dy/dx = 0$. Suppose $y = 4$, $y' = 0$. This rule directly follows from the power rule.

Derivatives of Composite Functions (Chain Rule)

If f and g are differentiable functions in their domain, then f(g(x)) is also differentiable. This is known as the chain rule of differentiation used for composite functions. $(fog)'(x) = f'[(g(x)]\, g'(x)$. This also can be write as "if $y = f(u)$ and $u = g(x)$, then $dy/dx = dy/du \cdot du/dx$.

For example, consider $y = \tan^2 x$. This is a composite function. We can write this function as $y = u^2$, where $u = \tan x$. Then

$dy/du = 2u$

$du/dx = d/dx (\tan x) = \sec^2 x$

By the chain rule,

$dy/dx = dy/du \cdot du/dx$

$= 2u \cdot \sec^2 x$

$= 2 \tan x \sec^2 x$

Derivatives of Implicit Functions

In equations where y as a function of x cannot be explicitly defined by the variables x and y, we use implicit differentiation. If $f(x, y) = 0$, then differentiate on both sides with respect to x and group the terms containing dy/dx at one side, and then solve for dy/dx.

For example, $2x + y = 12$

$d/dx(2x + y) = d/dx(0)$

$2 + dy/dx = 0$

$dy/dx = -2$

Parametric Derivatives

In a function, we may have the dependent variables x and y which are dependent on the third independent variable. If $x = f(t)$ and $y = g(t)$, then derivative is calculated as $dy/dx = f'(x)/g'(x)$. Suppose, if $x = 4 + t^2$ and $y = 4t^2 -5t^4$, then we find dy/dx as follows.

$dx/ dt = 2t$ and $dy/dt = 8t -20t^3$

$dy/dx = (dy/dt)/(dx/dt)$

$dy/dx = (8t -20t^3)/2t$

$= 2t (4 - 10t^2) / 2t$

$dy/dx = 4 - 10t^2$

Higher-order Derivatives

We can find the successive derivatives of a function and obtain the higher-order derivatives. If y is a function, then its first derivative is dy/dx. The second derivative is d/dx (dy/dx) which also can be written as d^2y/dx^2. The third derivative is $d/dx (d^2y/dx^2)$ and is denoted by d^3y/dx^3 and so on.

Alternatively, the first, second, and third derivatives of f(x) can be written as f'(x), f''(x), and f'''(x). For higher order derivatives, we write the number in brackets as the exponent. Suppose

$y = 4x^3$, we get the successive derivatives as follows. $y' = 12x^2$, $y'' = 24x$ and $y''' = 24$, $y^{(4)} = 0$.

Partial Derivatives

If $u = f(x,y)$ we can find the partial derivative of with respect to y by keeping x as the constant or we can find the partial derivative with respect to x by keeping y as the constant. Suppose $f(x, y) = x^3 y^2$, the partial derivatives of the function are:

- $\partial f/\partial x(x^3 y^2) = 3x^2 y$ and
- $\partial f/\partial y(x^3 y^2) = x^3 2y$

Further, we can find the second-order partial derivatives also like $\partial^2 f/\partial x^2$, $\partial^2 f/\partial y^2$, $\partial^2 f/\partial x\, \partial y$, and $\partial^2 f/\partial y\, \partial x$.

Finding Derivative Using Logarithmic Differentiation

Sometimes, the functions are too complex to find the derivatives (or) one function might be raised to another function like $y = f(x)^{g(x)}$. In such cases, we can take log (or) ln on both sides, apply log rules, and then differentiate on both sides to get dy/dx. This process is known as logarithmic differentiation in calculus.

$$\frac{d}{dx}.\log f(x) = \frac{f'(x)}{f(x)}$$

Example 2.1.1: Find the derivative of $y = x^x$.

Solution:

Applying ln on both sides,

$\ln y = \ln x^x$

$\ln y = x \ln x$

Taking derivative on both sides,

$1/y\ dy/dx = x\,(1/x) + \ln x\,(1)$ (by chain rule on left side and product rule on right side)

$1/y\ dy/dx = 1 + \ln x$

$dy/dx = y\,(1 + \ln x) = x^x\,(1 + \ln x)$

Maxima/Minima by Using Derivatives

The concept of slope, and hence the derivatives, is used to find the maximum or minimum value of a function. There are two tests that use derivatives and are used to find the maxima/minima of a function. They are

- first derivative test

- second derivative test

First Derivative Test

We can just use the first derivative to determine the maximum or minimum by observing (Figure 2.2.1) the following points:

- f'(x) represents the slope of a tangent line.
- Hence, if f'(x) > 0, the function is increasing, and if f'(x) < 0, the function is decreasing.
- If f'(x) > 0 is changing to f'(x) < 0 at a point, then the function has a local maximum at that point.
- If f'(x) < 0 is changing to f'(x) > 0 at a point, then the function has a minimum at that point.
- Note that f'(x) = 0 at local maximum and local minimum.

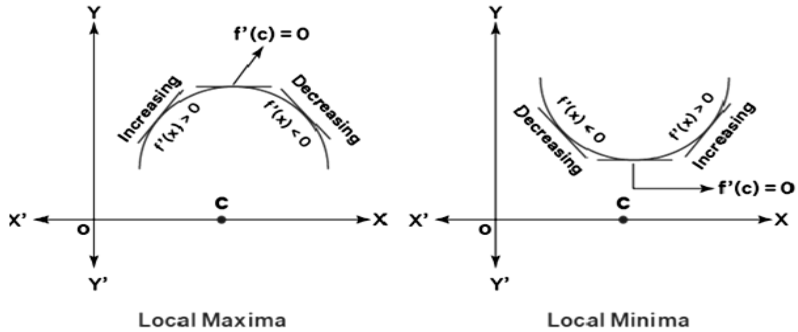
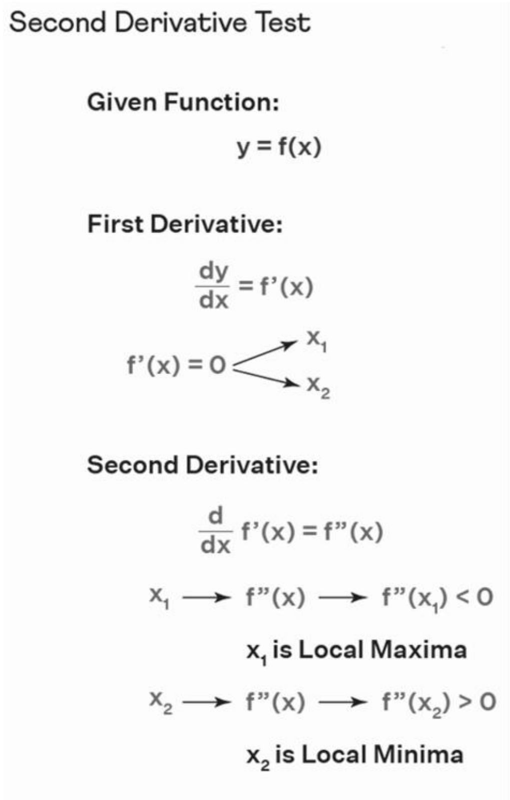


Figure 2.2.1 Maxima & Minima

Second Derivative Test

The second derivative test uses the critical points and the second derivative to find the maxima/minima. To perform this test:

- Find the critical points by setting f'(x) = 0.
- Substitute each of these in f''(x). If f''(x) < 0, then the function is maximum at that point and if f''(x)>0, then the function is minimum at that point.
- If f''(x) = 0, the function neither has maxima nor minima at that point, and in this case, it is known as the point of inflection.



2.3 Summary

The derivative of a function represents its rate of change with respect to its independent variable. It gives the slope of the tangent line to the graph of the function at any point.

- The derivative of a function f(x) and higher-order derivatives with respect to x is denoted by $f'(x), f''(x), f'''(x)$,......etc.
- **Rules an formulas**

Power Rule: $\frac{d}{dx}[x^n] = nx^{n-1}$.

Constant Rule: $\frac{d}{dx}[c] = 0$ for any constant c.

Sum and Difference Rule: $\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}[f(x)] \pm \frac{d}{dx}[g(x)]$.

Product Rule: $\frac{d}{dx}[f(x) \cdot g(x)] = f'(x) \cdot g(x) + f(x) \cdot g'(x)$.

Quotient Rule: $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{[g(x)]^2}$.

Chain Rule: If $y = f(u)$ and $u = g(x)$, then $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$.

Exponential and Logarithmic Functions: $\frac{d}{dx}[e^x] = e^x$ and $\frac{d}{dx}[\ln(x)] = \frac{1}{x}$.

Trigonometric Functions: $\frac{d}{dx}[\sin(x)] = \cos(x)$, $\frac{d}{dx}[\cos(x)] = -\sin(x)$, $\frac{d}{dx}[\tan(x)] = \sec^2(x)$, etc.

2.4 Keywords

- Calculus
- Derivatives
- Parametric Derivatives
- Higher Order Derivatives
- Maxima and Minima

2.5 Self – Assessment Questions

1. Differentiate $f(x)=3x^4-5x^2+2x-7$.
2. Differentiate $g(x)=(2x^3+3x^2-x+1)/x^2$.
3. Find the equation of the tangent line to $f(x)=x^2-4x+3$ at $x=2$.
4. Determine the local extrema of $h(x)=x^3-6x^2+9x+15$.
5. Find the intervals of concavity and the inflection points for $k(x)=x^4-8x^2$.

2.6 Case Study

A ladder 10 feet long is leaning against a wall. If the bottom of the ladder is sliding away from the wall at a rate of 1 foot per second, how fast is the top of the ladder sliding down the wall when the bottom is 6 feet from the wall?

2.7 References

1. Larson, R., & Edwards, B. (2017). Calculus of a Single Variable (11th ed.). Cengage Learning.
2. Stewart, J. (2016). Calculus: Early Transcendentals (8th ed.). Cengage Learning.

CHAPTER - 3
Limits and Continuous Function

Learning Objectives:

- Apply the concept of limits to analyze the behaviour of functions near specific points.
- Use algebraic manipulation and limit laws to simplify expressions and evaluate limits.
- Identify the three conditions for a function to be continuous at a point (existence of the limit, existence of the function value, equality of the limit and function value).

Structure:

3.1 Limits and Continuity: Definitions, Types, Formulas

In calculus, limits and continuity are important concepts that help us understand the behavior of functions as they approach certain values. A limit is a value that a function approaches as its input approaches a certain value. Continuity is the property of a function that describes whether it has any sudden jumps or breaks, or whether it can be drawn without lifting the pen from the paper.

Introduction to Limits and Continuity

- Limits and continuity are fundamental concepts in calculus. They allow us to understand how functions behave as their inputs approach certain values and to make precise statements about the behaviour of functions at certain points.
- Continuity refers to the property of a function where the output changes smoothly and gradually as the input changes. A function is continuous at a point in the event that the limit of the function at that point exists and is equal to the esteem of the function at that point. In the event that a function isn't continuous at a point, it is said to be discontinuous.
- Limits, on the other hand, allude to the behaviour of a function as the input approaches a certain value/point. A limit can be utilized to portray what happens to a function if the input gets subjectively near to a certain value, if the function isn't defined at that point.

Definition 3.1.1

The formal definition of a limit is given as follows:

For a function f(x), the limit of f(x) as x approaches a equals L, denoted as

Lim. f(x) = L

for every number $\varepsilon > 0$ there exists a corresponding number $\delta > 0$ such that $|f(x) - L| < \varepsilon$ whenever $0 < |x - a| < \delta$.

This definition can be hard to understand at first, but it basically says that as x gets arbitrarily close to a, f(x) gets arbitrarily close to L. The ε and δ values represent arbitrarily small positive numbers that determine the level of precision required for the limit.

For example, consider the function

$f(x) = (x^2 - 1)/(x - 1)$

If we try to evaluate f(1), we get an undefined expression (0/0). However, we can find the limit of f(x) as x approaches 1 by factoring the numerator and simplifying:

$= f(x) = (x^2 - 1)/(x - 1)$

$= (x + 1)(x - 1)/(x - 1)$

= x + 1

Now we can see that as x approaches 1, f(x) approaches 2. Therefore, we can write lim f(x) as x approaches 1 = 2.

Properties of Limits

There are several important properties of limits that are useful for evaluating and manipulating functions. These include:

- The limit of a whole is the sum of the limits:
  lim (f(x) + g(x)) = lim f(x) + lim g(x)
- The limit of a product is the product of the limits:
  lim (f(x) * g(x)) = lim f(x) * lim g(x)
- The limit of a quotient is the quotient of the limits:
  lim (f(x) / g(x)) = lim f(x) / lim g(x) (given lim g(x) ≠ 0)
- The limit of a constant multiple is the constant times the restrain:
  lim (k * f(x)) = k * lim f(x)
- The limit of a composite function is the restrain of the external function connected to the constrain of the inner function:
  lim f(g(x)) = f(lim g(x))

In expansion to these properties, there are a few hypotheses that can be utilized to assess limits, such as the squeeze theorem, which states that in the event that

$g(x) \leq f(x) \leq h(x)$

for all x in a few interim containing a, and lim g(x) = lim h(x) = L, at that point lim f(x) = L. Another important theorem is the intermediate value theorem, which states that if f is a continuous function on the closed interval [a, b] and c is a number between f(a) and f(b), at that point there exists a number x between a and b such that f(x) = c.

Let's outline a few of these properties with a case. Consider the work

$f(x) = x^2 - 3x + 2$

We can use the properties of limits to evaluate lim f(x) as x approaches 2.

First, we can factor in the expression:

$f(x) = x^2 - 3x + 2$

= (x - 1)(x - 2)

Using the product property of limits, we can write:

lim f(x) as x approaches 2 = lim (x - 1)(x - 2) as x approaches 2

Then, we can use the limit laws to simplify this expression:

= lim (x - 1)(x - 2) as x approaches 2

= (lim (x - 2)) * (lim (x - 1)) as x approaches 2

= (2 - 2) * (2 - 1)

= 0

Therefore, we can conclude that lim f(x) as x approaches 2 = 0.

Evaluating Limits

Several methods for evaluating limits include direct substitution, factoring, and L'Hopital's rule. Direct substitution is the simplest method, and it involves substituting the input value directly into the function and evaluating the resulting expression.

For example, to evaluate

lim $(x^2 - 4x + 3)/(x - 3)$

as x approaches 3, we can simply substitute x = 3 into the expression to get:

lim $(x^2 - 4x + 3)/(x - 3)$ = lim $(3^2 - 4(3) + 3)/(3 - 3)$ = lim 0/0

Since this expression is indeterminate, we can use factoring or L'Hopital's rule to evaluate the limit further.

Factoring involves rewriting the function as a product of simpler expressions that can be canceled out.

For example, using the same limit as above, we can factor the numerator as

(x - 3)(x - 1) and cancel out the common factor of (x - 3) to get:

lim $(x^2 - 4x + 3)/(x - 3)$ = lim (x - 1)

as x approaches 3

This limit evaluates to 2, so we can conclude that

lim $(x^2 - 4x + 3)/(x - 3)$ as x approaches 3 = 2

L-Hospital's rule is another method for evaluating indeterminate limits. It involves taking the derivative of both the numerator and denominator of the function and evaluating the resulting expression. For example, to evaluate lim sin(x)/x as x approaches 0, we can apply L-Hospital's rule to get:

lim sin(x)/x =lim cos(x) as x approaches 0

This limit evaluates to 1, so we can conclude that lim sin(x)/x as x approaches 0 = 1.

3.2 Continuity and Differentiability

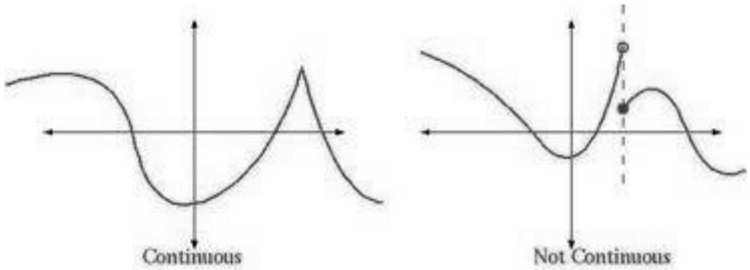


Figure 3.2.1 Continuity and Differentiability

- Continuity and differentiability are closely related concepts in calculus. A function is said to be continuous at a point if its graph has no breaks or jumps at that point. A function is said to be differentiable at a point if it has a well-defined tangent line at that point.
- The differentiability implies continuity theorem states that if a function is differentiable at a point, then it is also continuous at that point. However, the converse is not necessarily true - a function can be continuous at a point but not differentiable at that point.
- For example, the function $f(x) = |x|$ is continuous at $x = 0$, but it is not differentiable at that point because it has a sharp corner.
- The relationship between continuity and differentiability can also be communicated regarding the derivative. In the event that a function is differentiable at a point, at that point its derivative exists at that point.
- The derivative gives the slant of the tangent line at that point, and the slant of the tangent line can be utilized to decide whether the function is expanding or diminishing at that point.

Types of Discontinuities

- A discontinuity occurs when a function has a break in its graph, either due to a hole, jump, or asymptote.
- A removable discontinuity happens when a function has a gap in its graph, which can be filled in by rethinking the function at that point.
- For illustration, the function $f(x) = (x^2 - 4)/(x - 2)$ includes a removable discontinuity at $x = 2$, where the function is vague. Be that as it may, we are able to redefine the work as $f(x) = x + 2$ for $x \neq 2$ to fill within the gap and make the function nonstop.
- A jump discontinuity happens when a work has two diverse limits from the left and right sides of a point. For illustration, the work f(x) = {x on the off chance that $x < 0$, 1 in case x

≥ 0} includes a jump discontinuity at x = 0, since the left-hand restrain is and the right-hand constrain is 1.

- An infinite discontinuity occurs when a function approaches positive or negative infinity at a certain point. For example, the function $f(x) = 1/x$ has an infinite discontinuity at $x = 0$ since the function grows without bound as x approaches 0.

Applications of Limits and Continuity

Limits and continuity are important mathematical concepts that have a variety of applications in data science. Here are a few examples:

1. **Derivatives and gradients:** Limits are an essential component of calculus, which is the foundation of many data science techniques. Derivatives, which are based on limits, are used to calculate gradients, which are crucial for optimizing models using techniques such as gradient descent.
2. **Probability distributions:** Continuous probability distributions are often used in data science to model real-world phenomena. For example, the normal distribution is commonly used to model the distribution of heights or weights in a population.
3. **Regression analysis:** In regression analysis, a continuous variable is predicted based on other variables. The concept of continuity is essential here, as the prediction function must be continuous in order to be useful.
4. **Time series analysis:** Time series data is a sequence of values that are ordered by time. The continuity of time is an important aspect of time series analysis, as it allows us to model the relationship between past and future values.
5. **Signal processing:** Many signals, such as audio or video, are continuous in nature. Techniques from calculus, such as Fourier transforms, are used to analyze and process these signals.

Conclusion

In conclusion, limits and continuity are fundamental concepts in calculus that help us understand the behaviour of functions as they approach certain values. They are important in many fields, such as physics, economics, and engineering, and are used to model and optimize various systems. The relationship between continuity and differentiability is also important, with differentiability implying continuity. Overall, understanding limits and continuity is crucial for a solid foundation in calculus.

Key Takeaways

1. A limit is a value that a function approaches as its input approaches a certain value.
2. Continuity is the property of a function that describes whether it has any sudden jumps or breaks.
3. Evaluating limits can involve different methods, such as direct substitution, factoring, and L-Hospital's rule.
4. Types of discontinuities include removable, jump, and infinite discontinuities.
5. Differentiability implies continuity, but the converse is not necessarily true.
6. Limits and continuity are fundamental concepts in calculus and are important for understanding the behaviour of functions.

3.3 Differentiability of Functions:

The differentiation of a function gives the change of the function value with reference to the change in the domain of the function. Differentiability of a function can be understood both graphically and algebraically. Geometrically the differentiation of function is the slope of the graph of the function y = f(x) at the point x=a, in the domain of the function. Algebraically the differentiation of the function is the change in the value of the function y = f(x) from (x_1) to (x_2), with reference to the change in the domain value of x from x_1, to x_2. This can be expressed as $\frac{dy}{dx} = \frac{f(x_2)-f(x_2)}{x_2-x_1}$.

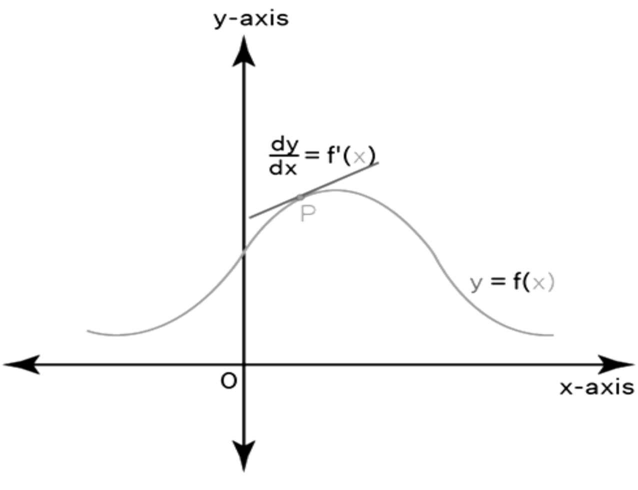


Figure 3.3.1: Differentiability of Functions

Differentiation of the function at point $P = \frac{dy}{dx}$

For a real valued function f(x) having a point x = c in the domain of this function, the derivative of the function f(x) at the point x = c is defined as $Lim_{x\to c} f^{'}(c) = \frac{f(c+h)-f(c)}{h}$. Thus the derivative of a function is defined as f'(x) or (d/dx)f(x) and is also represented as $f^{'}(x) = Lim_{x\to h} \frac{f(x+h)-f(x)}{h}$. This process of finding the derivative is called as differentiation. Also, the phrase differentiate f(x) with respect to x is used to mean dy/dx or f'(x).

The three important rules of the algebra of differentiation of functions are as follows.

(f + g)'(x) = f'(x) + g'(x)

(f.g)'(x) = f'(x).g(x) + g'(x).f(x)

(f/g)'(x) = (f'(x).g(x) - g'(x).f(x))/$(f(x))^2$

The following are some of the important differentiation of the functions f(x) based on the type of functions.

- Derivative of Composite Function
- Derivatives of Implicit Functions
- Derivatives of Inverse Trigonometric Functions
- Derivatives of Exponential Functions
- Derivatives of Logarithmic Functions
- Derivatives of Functions in Parametric Forms

Theorems on Continuity and Differentiability

The following important theorems on continuity and differentiability, set the right background for the deeper understanding of the concepts of continuity and differentiability.

Theorem 3.3.1: If two functions f(x) and g(x) are continuous at a real valued function and continuous at a point x = c, then we have:

f(x) + g(x) is continuous at the point c = c

f(x) - g(x) is continuous at a point x = c

g(x).g(x) is continuous at point x = c

f(x)/g(x) is continuous at a point x = c, provided g(c) ≠ 0

Theorem 3.3.2: For two real values functions f(x) and g(x) such that the composite function fog(x) is defined at x = c. If g(x) is continuous at x = c and the function f(x) is continuous at g(c), then fog(x) is continuous at x = c.

Theorem 3.3.3: If a given function f(x) is differentiable at a point x = c, then it is continuous at that point. This can be summarized as every differentiable function is continuous.

Theorem 3.3.4: Chain Rule: For a real valued function f(x), which is a composite of two functions u and v ie, f = vou. Also let us suppose t = u(x) and if both dt/dx and dv/dt exists, then we have df/dx = dv/dt.dt.dx.

Theorem 3.3.5: The derivative of e^x with respect to x is e^x. d/dx.e^x = 1. And the derivative of logx with respect to x is 1/d. d/dx. logx = 1/x.

Theorem 3.3.6: (Rolle's Theorem). If a function f(x) is continuous across the interval [a, b] and differentiable across the interval (a, b), such that f(a) = f(b), and a, b are some real numbers. Then there exists a point c in the interval [a, b] such that f'(c) = 0.

Theorem 3.3.7: (Mean Value Theorem). If a function f(x) is continuous across the interval [a, b] and differentiable across the interval (a, b), then there exists a point c in the interval [a, b] such that $f'(c)=f(b)-f(a)b-a$.

Successive Differential

Successive differentiation is the differentiation of a function successively to derive its higher-order derivatives.

1. If $y=f(x)$ is a function of x, then the derivative of y with respect to x is denoted by dy/dx or Dy or $f'(x)$ or y_1. This is the first-order derivative of y.
2. If dy/dx is differentiated again, $y=f(x)$ is derivable twice with respect to x, then the derivative of dy/dx with respect to x is denoted by d^2y/dx^2 or D^2y or $f''(x)$ or y_2.This is the second-order derivative of y.
3. If d^2y/dx^2 is differentiated again, $y=f(x)$ is derivable thrice with respect to x, then the derivative of d^2y/dx^2 with respect to x is denoted by d^3y/dx^3 or D^3y or $f'''(x)$ or y_3. This is called the third-order derivative of y.

Implicit vs Explicit

A function can be explicit or implicit:

Explicit: "y = some function of x". When we know x we can calculate y directly.

Implicit: "some function of y and x equals something else". Knowing x does not lead directly to y.

Example 3.3.1:

Explicit Form	**Implicit Form**
$y = \pm \sqrt{(r^2 - x^2)}$	$x^2 + y^2 = r^2$
In this form, y is expressed as a function of x.	In this form, the function is expressed in terms of both y and x.

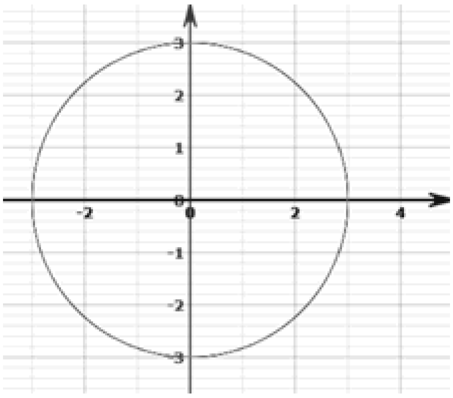


Figure 3.3.2: The graph of $x^2 + y^2 = 3^2$

How to do Implicit Differentiation

- Differentiate with respect to x
- Collect all the *dy/*dx on one side
- Solve for *dy/*dx

Example 3.3.1: $x^2 + y^2 = r^2$

Differentiate with respect to x:

$d/dx(x^2) + d/dx(y^2) = d/dx(r^2)$

Let's solve each term:

Use the Power Rule:$d/dx(x^2) = 2x$

Use the Chain Rule (explained below):$d/dx(y^2) = 2y dy/dx$

r^2 is a constant, so its derivative is 0:$d/dx(r^2) = 0$

Which gives us:

$2x + 2y dy/dx = 0$

Collect all the dy/dx on one side

ydy/dx = −x

Solve for dy/dx:

dy/dx = −xy

Leibniz Theorem

Leibniz rule generalizes the product rule of differentiation. The Leibniz rule states that if two functions f(x) and g(x) are differentiable n times individually, then their product f(x).g(x) is also differentiable n times. The Leibniz rule is $(f(x).g(x))^n = \sum^n C_r f^{(n-r)}(x).g^r(x)$

The Leibniz rule can be applied to the product of multiple functions and for numerous derivatives. Let us understand the different formulas and proof of Leibniz rule.

$$(f(x).g(x))^n = \sum^n C_r f^{(n-r)}(x).g^r(x)$$

Here n$C_r = \frac{n!}{r!.(n-r)!}$. and n! = 1 × 2 × 3 × 4 × ....(n - 1) × n.

The Leibniz rule is primarily used as the derivative of the product of two functions. The Leibniz rule for the first derivative of the product of the functions f(x), and g(x) is equal to the sum of the product of the first derivative of the function f(x), and the function g(x), and the product of the first derivative of the function g(x), and the function f(x). .

(f(x).g(x))' = f'(x).g(x) + f(x).g'(x)

or

$$\frac{d}{dx}.f(x).g(x) = g(x).\frac{d}{dx}.f(x) + f(x).\frac{d}{dx}.g(x)$$

Further we can use the leibniz rule to find the second derivative of the product of two functions. The leibniz rule for the second derivative of the product of the functions f(x) and g(x) can be understood to be similar to the binomial expansion of two terms of second degree. Let us also check the second derivative of the product of the functions.

(f(x).g(x))'' = f''(x).g(x) + 2f'(x).g'(x) + f(x).g'(x)

or

$$\frac{d^2}{dx^2}.f(x).g(x) = g(x).\frac{d^2}{dx^2}.f(x) + 2\frac{d}{dx}.f(x).\frac{d}{dx}.g(x) + f(x).\frac{d^2}{dx^2}.g(x)$$

The Leibniz rule can be applied to the product of multiple functions and for numerous derivatives. The Leibniz rule can be proved using mathematical induction.

Proof of Leibniz Rule

The Leibniz rule can be proved with the help of mathematical induction. Let f(x) and g(x) be n times differentiable functions. Applying the initial case of mathematical induction for n = 1 we have the following expression.

(f(x).g(x))' = f'(x).g(x) + f(x).g'(x)

Which is the simple product rule and it holds true for n = 1. Let us assume that this statement is true for all n $\geq$ 1, and we have the below expression.

$$(f(x).g(x))^n = \sum^{n} \quad C_r f^{(n-r)}(x).g^r(x)$$

Further we have the following expression for n + 1.

$$(f(x).g(x))^{n+1} =^n \sum_{r=0} C_r^n f^{(n-r)}(x).g^r(x)$$

$$=^n \sum_{r=0} .^n C_r.f^{(n+1-r)}(x).g^r(x) +^n \sum_{r=0} .^n C_r.f^{(n-r)}(x).g^{r+1}(x)$$

$$=^n \sum_{r=0} .^n C_r.f^{(n+1-r)}(x).g^r(x) +^{n+1} \sum_{r=1} .^n C_{r-1}.f^{(n+1-r)}(x).g^r(x)$$

$$=^n C_0.f^{n+1}(x).g(x) +^n \sum_{r=1} .^n C_r.f^{(n+1-r)}(x).g^r(x)$$

$$+^n \sum_{r=1} .^n C_{r-1}.f^{(n+1-r)}(x).g^r(x) +^n C_n.f(x).g^{n+1}(x)$$

$$= f^{n+1}(x).g(x) +^n \sum_{r=1} [^n C_r +^n C_r].f^{(n+1-r)}(x).g^r(x) + f(x).g^{n+1}(x).$$

$$= f^{n+1}(x).g(x) +^n \sum_{r=1} .^{n+1} C_r.f^{(n+1-r)}(x).g^r(x) + f(x).g^{n+1}(x)$$

$$=^{n+1} \sum_{r=0} .^{n+1} C_r.f^{(n+1-r)}(x).g^r(x)$$

Examples on Leibniz Rule

Example 3.3.2: Find the derivative of the product of the functions f(x) = x^4 , and g(x) = Logx, using the Leibniz rule.

Solution:

The two given functions are f(x) = x^4 , and g(x) = Logx.

The rule for Leibniz formula for product of two functions f(x), g(x) is

$$\frac{d}{dx}.f(x).g(x) = g(x).\frac{d}{dx}.f(x) + f(x).\frac{d}{dx}.g(x)$$

$$= \frac{d}{dx}.x^4.Logx = Logx.\frac{d}{dx}.x^4 + x^4.\frac{d}{dx}.Logx$$

$$= \frac{d}{dx}.x^4.Logx = Logx.4x^3 + x^4.\frac{1}{x}$$

$$\frac{d}{dx}.x^4.Logx = 4x^3.Logx + x^3$$

Therefore, by using leibniz rule the derivative of the product of the two given functions is $4x^3.Logx + x^3$.

3.4 Summary

Understanding limits and continuous functions is fundamental in the study of differential equations, as it lays the groundwork for analyzing and solving these equations accurately.

- **L'Hôpital's Rule**: If the limit results in an indeterminate form 0/0 or ∞/∞, differentiate the numerator and the denominator and then find the limit.
- **Limits:** The limit of a function f(x) as x approaches a point a is denoted as $\lim_{x \to a} f(x)$. It represents the value that f(x) gets closer to as x gets closer to a.
- **Continuous Functions:** A function f(x) is continuous at a point a if the following three conditions are met:
  1. f(a) is defined.
  2. $\lim_{x \to a} f(x)$ exists.
  3. $\lim_{x \to a} f(x) = f(a)$.
- A function f(x) is continuous on an interval if it is continuous at every point within the interval.

3.5 Keywords

- Limit and Continuity
- Differentiability
- Leibnitz's rule
- Euler's theorem

3.6 Self – Assessment Questions

1. Determine if $f(x)=(x^2-4)/(x+2)$ is continuous at $x=-2$.

2. Investigate the continuity of $g(x)=x$ at $x=0$.
3. Compute the limit of $f(x)=(x^2-1)/(x-1)$ as x approaches 1.
4. Find $\lim_{x\to\infty}(1/x)$ and $\lim_{x\to 0+}\ln(x)$.
5. Determine $\lim_{x\to\infty}(2x^2-3x+1)/(3x^2+5)$.
6. Calculate $\lim_{x\to-\infty}(e^x/x)$.
7. Apply L'Hôpital's Rule to evaluate $\lim_{x\to 0}(\sin(x)/x)$.

3.7 Case Study

a) The height of a ladder above the ground is dropping at a rate of 0.1 meters per second as it leans against a wall. With its base three meters from the wall, how quickly is the ladder slipping away from the wall?

b) What does the ladder's motion indicate from the pace at which its distance from the wall changes?

3.8 References

1. Larson, R., & Edwards, B. (2017). Calculus of a Single Variable (11th ed.). Cengage Learning.
2. Stewart, J. (2016). Calculus: Early Transcendentals (8th ed.). Cengage Learning.

CHAPTER - 4

Partial Differentiation

Learning Objectives:

- Apply the concept of partial derivatives to analyze the rate of change of a function with respect to one of its variables while holding other variables constant.
- Differentiate functions of multiple variables with respect to one variable while treating other variables as constants.
- Identify and analyze real-world applications of partial derivatives, such as optimization problems in economics, physics, and engineering.

Structure:

4.1 Partial Derivative

The partial derivative of a function (in two or more variables) is its derivative with respect to one of the variables keeping all the other variables as constants. The process of calculating partial derivative is as same as that of an ordinary derivative except we consider the other variables as the variable with respect to which we are differentiating as constants.

Let us learn more about how to calculate partial derivatives of different orders along with examples.

What is Partial Derivative?

The partial derivative of a multivariable function, say z = f(x, y), is its derivative with respect to one of the variables, x or y in this case, where the other variables are treated as constants. For example,

- for finding the partial derivative of f(x, y) with respect to x (which is represented by $\partial f / \partial x$), y is treated as constant
- for finding the partial derivative of f(x, y) with respect to y (which is represented by $\partial f / \partial y$), x is treated as constant

Note that we are not considering all the variables as variables while doing partial differentiation (instead, we are considering only one variable as a variable at a time) and hence the name "partial". The limit definition of a partial derivative looks very similar to the limit definition of the derivative. We can find the partial derivatives using the following limit formulas:

If z=f(x, y), then

- $\partial f / \partial x = \lim_{h \to 0} [\, f(x + h, y) - f(x, y) \,] / h$
- $\partial f / \partial y = \lim_{h \to 0} [\, f(x, y + h) - f(x, y) \,] / h$

These formulas resemble the derivative definition using the first principle.

Example 4.1.1

If f (x, y) = xy, then find the partial derivative $\partial f / \partial x$.

Solution:

$\partial f / \partial x = \lim_{h \to 0} [\, f(x + h, y) - f(x, y) \,] / h$

$= \lim_{h \to 0} [\, (x + h)y - xy \,] / h$

$= \lim_{h \to 0} [xy + hy - xy] / h$

$= \lim_{h \to 0} [hy]/h$

$= \lim_{h \to 0} y$

$= y$

Therefore, $\partial f / \partial x = y$.

Partial Derivative Symbol

We know that the ordinary derivative of a function y = f(x) is denoted by one of the notations dy/dx, d/dx (y), d/dx (f(x)), f '(x), etc. For representing a partial derivative, we use the symbol "∂" instead of "d". We pronounce "∂" to be "doh" but it has some other names like "partial", "del", "partial dee", "dee", "Jacobi's delta", etc. If z = f(x, y) is a function in two variables then

- $\partial f / \partial x$ is the partial derivative of f with respect to x
- $\partial f / \partial y$ is the partial derivative of f with respect to y

Just like how we have different symbols of ordinary derivatives, we have different notations for partial derivatives as well. For example, $\partial f / \partial x$ can be written as f_x, f_x', $D_x f$, $\partial / \partial x$ (f), ∂_x f, $\partial / \partial x$ [f(x, y)], $\partial z / \partial x$, etc.

Calculate Partial Derivatives

We have already seen that the limit definitions are used to find the partial derivatives. But using the limit formula and computing the limit is not always easy. Thus, we have another method to calculate partial derivatives that follow right from its definition. In this method, if z = f(x, y) is the function, then we can compute the partial derivatives using the following steps:

- **Step 1**: Identify the variable with respect to which we have to find the partial derivative.
- **Step 2:** Except for the variable found in **Step 1**, treat all the other variables as constants.
- **Step 3:** Differentiate the function just using the rules of ordinary differentiation.

Wait! Read **Step 3** again. Yes, the rules of ordinary differentiation are as same as that of partial differentiation. In partial differentiation, just treating variables is different, that's it!

Example 4.1.2

Let us solve the same above example (If f (x, y) = xy, then find the partial derivative $\partial f / \partial x$) using the above steps.

Solution:

We have to find $\partial f / \partial x$. It means, we have to find the partial derivative of f with respect to x. So we treat y as constant. Thus, we can write 'y' outside the derivative (as in ordinary differentiation, we have a rule that says d/dx (c y) = c dy/dx, where 'c' is a constant). Thus,

$\partial f / \partial x = \partial / \partial x\ (xy)$

$= y\ \partial / \partial x\ (x)$

$= y\ (1)$ (Using power rule, d/dx (x) = 1)

$= y$

We have got the same answer as we got using the limit definition.

Partial Derivatives of Different Orders

We have derivatives like first-order derivatives (like dy/dx), second-order derivatives (like d^2y/dx^2), etc in ordinary derivatives. Likewise, we have first-order, second-order, and higher-order derivatives in partial derivatives also.

4.2 First Order Partial Derivatives

If z = f(x, y) is a function in two variables, then it can have two first-order partial derivatives, namely $\partial f / \partial x$ and $\partial f / \partial y$.

Example 4.2.1

If $z = x^2 + y^2$, find all the first order partial derivatives.

Solution:

$f_x = \partial f / \partial x = (\partial / \partial x)\ (x^2 + y^2)$

$= (\partial / \partial x)\ (x^2) + (\partial / \partial x)\ (y^2)$

$= 2x + 0$ (as y is a constant)

$= 2x$

$f_y = \partial f / \partial y = (\partial / \partial y)\ (x^2 + y^2)$

$= (\partial / \partial y)\ (x^2) + (\partial / \partial y)\ (y^2)$

$= 0 + 2y$ (as x is a constant)

$= 2y$

4.3 Second Order Partial Derivatives

The second-order partial derivative is obtained by differentiating the function with respect to the indicated variables successively one after the other. If z = f(x, y) is a function in two variables, then it can have four second-order partial derivatives, namely $\partial^2 f / \partial x^2$, $\partial^2 f / \partial y^2$, $\partial^2 f / \partial x\ \partial y$ and $\partial^2 f / \partial y\ \partial x$. To find them, we can first differentiate the function partially with the latter variable, and then partially differentiate the result with respect to the former variable. i.e.,

- $f_{xx} = \partial^2 f / \partial x^2 = \partial / \partial x (\partial f / \partial x) = \partial / \partial x (f_x)$
- $f_{yy} = \partial^2 f / \partial y^2 = \partial / \partial y (\partial f / \partial y) = \partial / \partial x (f_y)$
- $f_{yx} = \partial^2 f / \partial x \partial y = \partial / \partial x (\partial f / \partial y) = \partial / \partial x (f_y)$
- $f_{xy} = \partial^2 f / \partial y \partial x = \partial / \partial y (\partial f / \partial x) = \partial / \partial y (f_x)$

Observe the notations f_{yx} and f_{xy}. The order of variables in each subscript indicates the order of partial differentiation. For example, f_{yx} means to partially differentiate with respect to y first and then with respect to x, and this is same as $\partial^2 f / \partial x \partial y$.

Example 4.3.1

If $z = x^2 + y^2$, find all the second order partial derivatives.

Solution:

In the above example, we have already found that $f_x = 2x$ and $f_y = 2y$.

Now, $f_{xx} = \partial / \partial x (f_x) = \partial / \partial x (2x) = 2$

$f_{yy} = \partial / \partial y (f_y) = \partial / \partial y (2y) = 2$

$f_{yx} = \partial / \partial x (f_y) = \partial / \partial x (2y) = 0$

$f_{xy} = \partial / \partial y (f_x) = \partial / \partial y (2x) = 0$

Now that $f_{yx} = f_{xy}$. Thus, the order of partial differentiation doesn't matter.

Partial Differentiation Formulas

The process of finding partial derivatives is known as Partial Differentiation. To find the first-order partial derivatives (as discussed earlier) of a function $z = f(x, y)$ we use the following limit formulas:

- $\partial f / \partial x = \lim_{h \to 0} [f(x + h, y) - f(x, y)] / h$
- $\partial f / \partial y = \lim_{h \to 0} [f(x, y + h) - f(x, y)] / h$

But instead of using these formulas, just treating all the other variables than the variable with respect to which we are partially differentiating as constants would make the process of partial differentiation very easier. In this process, we just use the same rules as ordinary differentiation and among them; the important rules are as follows:

4.4 Power Rule

The power rule of differentiation says $(d/dx0(x^n) = nx^{n-1}$. The same rule can be applied in partial derivatives also.

Example 4.4.1

$\partial / \partial x\ (x^2y)=?$

Solution:

$\partial / \partial x\ (x^2y) = y\ \partial / \partial x\ (x^2) = y\ (2x) = 2xy.$

4.5 Product Rule

The product rule of ordinary differentiation says d/dx (uv) = u dv/dx + v du/dx. We can apply the same rule in partial differentiation as well when there are two functions of the same variable.

Example 4.5.1

$\partial / \partial x\ (xy \sin x)=?$

Solution:

$\partial / \partial x\ (xy \sin x) = y\ \partial / \partial x\ (x \sin x)$

$= y\ [\ x\ \partial / \partial x\ (\sin x) + \sin x\ \partial / \partial x\ (x)]$

$= y\ [\ x \cos x + \sin x]$

4.6 Quotient Rule

The quotient rule of ordinary differentiation says d/dx (u/v) = [v du/dx - u dv/dx] / v^2. As other rules, this rule can be applied for finding partial derivatives also.

Example 4.6.1

$\partial / \partial x\ (xy / \sin x)=?$

Solution:

$\partial / \partial x\ (xy / \sin x)$

$= y\ \partial / \partial x\ (x / \sin x)$

$= y\ [\ (\ \sin x\ \partial / \partial x\ (x) - x\ \partial / \partial x\ (\sin x)\ ) / \sin^2 x\ ]$

$= y\ [\ \sin x - x \cos x] / \sin^2 x$

4.7 Chain Rule of Partial Differentiation

The chain rule is used when we have to differentiate an implicit function. The chain rule of partial derivatives works a little differently when compared to ordinary derivatives. Sometimes, the rule involves both partial derivatives and ordinary derivatives. There are various forms of this rule and the application of one of them depends upon how each variable of the function is defined.

- If y = f(x) is a function where x is again a function of two variables u and v (i.e., x = x (u, v)) then
  $\partial f/\partial u = \partial f/\partial x \cdot \partial x/\partial u$;
  $\partial f/\partial v = \partial f/\partial x \cdot \partial x/\partial v$
- If z = f(x, y), where each of x and y are again functions of a variable t (i.e., x = x(t) and y = y(t)) then
  $df/dt = (\partial f/\partial x \cdot dx/dt) + (\partial f/\partial y \cdot dy/dt)$
- If z = f(x, y) is a function and each of x and y are again functions of two variables u and v (i.e., x = x(u, v) and y = y(u, v)) then
  $\partial f/\partial u = \partial f/\partial x \cdot \partial x/\partial u + \partial f/\partial y \cdot \partial y/\partial u$;
  $\partial f/\partial v = \partial f/\partial x \cdot \partial x/\partial v + \partial f/\partial y \cdot \partial y/\partial v$

Example 4.7.1

If $z = e^{xy}$, where x = uv and y = u + v then find the partial derivative $\partial f/\partial u$.

Solution:

By the chain rule of partial derivatives:

$\partial f/\partial u = \partial f/\partial x \cdot \partial x/\partial u + \partial f/\partial y \cdot \partial y/\partial u$

$= \partial / \partial x\,(e^{xy}) \cdot \partial / \partial u\,(uv) + \partial / \partial y\,(e^{xy}) \cdot \partial / \partial u\,(u + v)$

$= (e^{xy} \cdot y)\,(v) + (e^{xy} \cdot x)\,(1)$

$= e^{xy}\,(x + vy)$

Other Rules of Partial Differentiation

- If f(x, y) = a constant, then the following formula gives the relation between the ordinary derivative and the partial derivatives which follows from implicit differentiation. $dy/dx = -f_x/f_y$.
- For any two functions u(x, y) and v(x, y), the determinant |||(∂u/∂x)(∂u/∂y)(∂v/∂x)(∂v/∂y)||||$(\partial u/\partial x)(\partial u/\partial y)(\partial v/\partial x)(\partial v/\partial y)$| is known as **Jacobian** of u and v.
- The **Laplace equation** of partial derivatives is $\partial^2 f/\partial x^2 + \partial^2 f/\partial y^2 + \partial^2 f/\partial z^2 = 0$ where f(x, y, z) is a function in three variables. Any function f that satisfies the Laplace equation is known as the harmonic function.

4.8 Maxima and Minima

Maxima and Minima are called critical points of the function. A maxima is a high point and a minima is a low point in any function. In a function, more than one maximum and minimum point can exist. The points at which the function attains the highest and lowest values are called Maxima and Minima.

Relative Maxima and Minima

Relative maxima and minima are the points at which the function gives the maximum and minimum values respectively in their neighbourhood. Relative maxima and minima of any function are easily found by using the first derivatives and second derivative test respectively. The graph added below the relative maxima and minima (Figure 4.8.1) of a function in its neighbourhood.

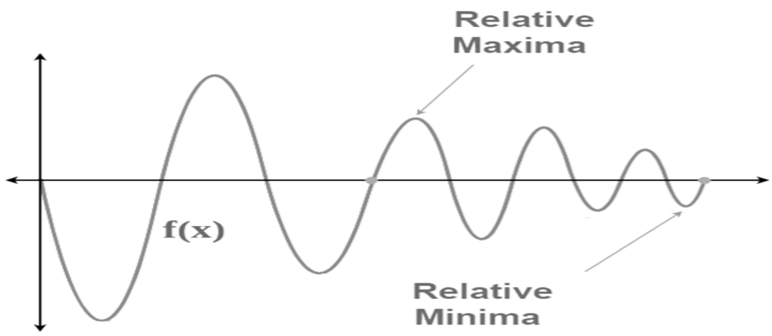


Figure 4.8.1 Relative Maxima & Minima

Points of Maxima and Minima (First Derivative Test)

In any smoothly changing function, the points where the function flattens out, give us either minima or maxima. Now, this statement gives rise to two questions.

1. How to recognize the points at which function flattens out?
2. Suppose we got a point at which function flattens i.eofcritical point. How to tell whether it’s a minimum or a maximum?

To answer the first question let’s look at the slope of the function. The points where the function flattens out have zero slopes. We know that the derivative is nothing but the slope of the function at a particular point. So, we try to find the points where the derivative is zero. Thus, this test is also called the First Derivative Test. Then we equate the differential equation with zero to get the critical points as,

$f'(x) = 0$

The solution to this equation gives us the position of the critical points. These critical points tells us that these are the points where the tangent to the curve is parallel to x-axis but still we don't know whether they are points o maxima or minima for that Second Derivative test is used.

Recognizing Maxima and Minima

As shown in the figure below, it can be seen that if the sign of the derivative is positive before the critical point and negative after the critical point, it is a maximum. Similarly, if it is negative before the critical point and positive after the critical point. It is a minimum. Maxima and minima can also be recognized by the second derivative test.

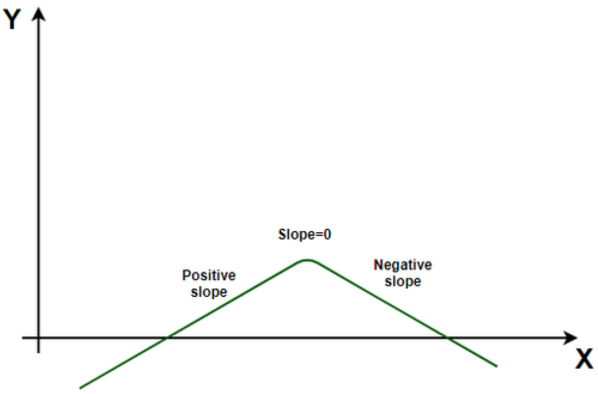


Figure 4.8.2 Maxima & Minima Slope

Notice the Figure 4.8.2 carefully, and see that the slope of the curve is continuously decreasing, and then it becomes zero and goes further towards a negative value.

Second Derivative Test

When a function's slope is zero at x, then the second derivative f''(x) at that point is used to tell whether at that point we have maxima or minima.

1. If $f''(x) < 0$, then maxima.
2. If $f''(x) > 0$, then minima.

Note: If the second derivative is zero at the critical points, then the test fails.

Steps to Find Relative Maxima and Minima

To find the relative maxima and minima of a function follow the steps added below, for a given function,

f(x) = x2 – 4

We obtain the maximum and minimum value in the interval [-2, 2] as,

Step 1: Differentiate the Function

Given,

- $f(x) = x^2 - 4$
- $f'(x) = 2x$

Step 2: Find out Critical Points

Putting $f'(x) = 0$ gives us the position of the critical points for the function. For the given function,

$2x = 0$

$\Rightarrow x = 0$

Thus, $x = 0$ is a critical point for this function. Now we need to find out whether it is a minimum or a maximum.

Step 3: Test for the Second Derivative

We use the second derivative test mentioned above to obtain whether the given critical point is minima or maxima. In the above case,

$f''(x) = 2$

Notice that, $f''(x) > 0$. Thus, it must be a minimum.

Step 4: Value at Critical Points

Find the value of the function at critical points get the smallest value(or the minima)

- f(0) = (0)2 – 4 = -4

Thus, the minimum value of f(x) is at x = 0 and the minimum value is -4.

Applications of Relative Maxima and Minima

Relative Maxima and Minima has various applications. it is used for various purposes such as,

- This concept is used to determine the maximum and minimum value of a stock at particular points by equating it to a function that represent the trajectory of the stocks.
- This concept are used in electronic circuits to manage the voltage and current in the system.

- Relative maxima and minima is also used in astrophyscis to find the maximum and minimum trajectory of an abject, etc.

Absolute Minima and Maxima

Absolute Maxima and Minima are the maximum and minimum value of the function defined in fixed interval. A function in general can have high values or low values as we move along the function. The maximum value of function in any interval is known as the maxima and minimum value of function is named as the minima. These maxima and minima if defined on the whole functions are called the Absolute Maxima and Absolute Minima of the function.

In this article, we will learn about Absolute Maxima and Minima, How to calculate absolute maxima and minima, their examples, and others in detail.

Definition 4.8.1

Absolute maximum and absolute minimum

An **absolute maximum** of $f$at c if $f(c) \geq f(x)$for all x in domainof f. The number $f(c)$ is called maximum value of$f$ in the domain. Similarly an **absolute minimum** of $f$at c if $f(c) \leq f(x)$ for all x in domain of f and the number $f(c)$ is called the minimum value of f on the domain. The maximum and minimum value of f are called extreme values of f.

Absolute maxima and minima are maximum and minimum value of function on entire given range. Absolute Maxima and Minima are similarly named the global maxima and minima of the function it is the maximum and the minimum value that the function can achieve in its entire domain. Suppose we have a function f(x) = sinx , defined on interval R that **is** $-1 \leq \sin x \leq 1$.

Thus,

- Maximum value of f(x) is 1
- Minimum value of f(x) is -1

Thus, absolute maxima and minima of f(x) = sin x defined over R is 1 and -1.

Critical Points and Extrema Value Theorem

Let's say we have a function f(x), critical points are the points with derivative of the function becomes zero. These points can either be maxima or minima. A critical point is minima or maxima are determined by the second derivative test. Since there can be more than one point where the derivative of the function is zero, more than minima or maxima is possible. The figure below shows a function that has multiple critical points.

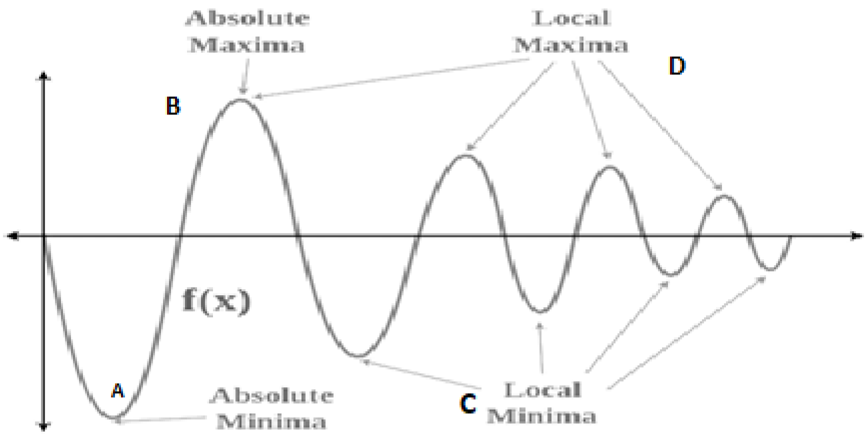


Figure 4.8.3 Absolute and Local Points of the Maxima & Minima

Notice the Figure 4.8.3, that points A, C are minima, and points B, D are maxima. B and C are called local maxima and local minima respectively. This means that these points are maximum and minimum in their locality but not necessarily on a global level. Points A and D are called global minima and global maxima.

Let's say we have a function f(x) which is twice differentiable. Its critical points are given by the f'(x) = 0. Second Derivative Test allows us to check whether the calculated critical point is minima or maxima.

- If f''(x) > 0, then the point x is maxima.
- If f''(x) < 0, then the point x is minima.

Now, this test tells us which point is a minimum or a maximum, but it still fails to give us information about the global maxima and global minima. Extrema value Theorem comes to our rescue.

Extrema Value Theorem

Extrema value theorem guarantees both the maxima and minima for a function under certain conditions. This theorem does not tell us where the extreme points will exist, this theorem tells us only that extreme value will exist. The theorem states that,

If, function f(x)is continuous on closed interval [a, b], then f(x) has both at least one maximum and minimum value on [a, b].

Absolute Minima and Maxima in Closed Interval

Now to find the extreme points in any interval, we need to follow some basic steps. Let’s say we have a function f(x) and a region D. We obtain the extreme value of the function in the interval.

Step 1: Find the critical points of function in interval D,

$f'(x) = 0$

Step 2: Find the value of the function at the extreme points of interval D.

Step 3: The largest value and smallest value found in the above two steps are absolute maximum and absolute minimum of function.

Absolute Minima and Maxima in Entire Domain

Absolute minimum and maximum values of function in the entire domain are the highest and lowest value of the function wherever it is defined. A function can have both maximum and minimum values, either one of them or neither of them. For example, a straight line extends up to infinity in both directions so it neither has a maximum value nor minimum value.

We need to follow some steps similar to the previous case to find out the absolute maxima and minima for the entire domain.

Step 1: Find the critical points of the function wherever it is defined.

Step 2: Find the value of the function at these extreme points.

Step 3: Check for the value of the function when x tends to infinity and negative infinity. Also, check for the points of discontinuity.

Step 4: Maximum and minimum of all these values give us the absolute maximum and absolute minimum for the function in its entire domain.

Local Maxima and Minima

Local Maxima and Local Minima are the maximum and minimum value of the function relative to other points over a specific interval of the function. They are generally calculated in the same way we calculate Absolute maxima and minima. Local maxima and minima of any function can be similar or not similar to Absolute maxima and minima of the function.

Suppose we have a function $f(x) = \cos x$ defined on $[-\pi, \pi]$ then is maximum value is 1 and its minimum value is -1 this is the local maxima and minima of the function. Now the function f(x) defined on R also has the maximum and minimum value of the function to be 1 and -1 this is absolute maxima and minima of the f(x). Here, we can see that local maxima and minima of the function are similar.

- f functionhas a local maximum (or relative maximum) at c if an open interval (a,b) containing c such that $f(c) \geq f(x)$ for every $x \in (a,b)$
- Similarly,local minimum of f at c if there is an open interval (a,b) containing c such that $f(c) \leq f(x)$ for every x ! (a,b).

Note

Absolute maximum and absolute minimum values of a function f on an interval (a,b) are also called the global maximum and global minimum of f in (a,b).

Criteria for local maxima and local minima

Let f be a differentiable function on an open interval (a,b) containing c and suppose that $f''(c)$ exists.

(i) If $f'(c) = 0$ and $f''(c) > 0$, then local minimum of $f$at c.

(ii) If $f'(c) = 0$ and $f''(c) < 0$,then local maximum of $f$at c.

Note

In Economics, if $y = f(x)$ represent cost function or revenue function, then the point at which $dy/dx = 0$, the cost or revenue is maximum or minimum.

Example 4.8.1

Find extrema values of $f(x)=2x^3+3x^2-12x$.

Solution:

Given

$f(x) = 2x^3+3x^2-12x$ … (1)

$f'(x) = 6x^2+6x-12$

$f''(x) = 12x + 6$

$f'(x) = 0 \Longrightarrow 6x^2+6x-12 = 0$

$\Longrightarrow 6(x^2+x-2)= 0$

$\Longrightarrow 6(x+2)(x-1)= 0$

$\Longrightarrow x = -2$ and $x = 1$

When

$x= -2$

$f''(-2) = 12(-2) + 6$

$= -18 < 0$

$f(x)$ attains local maximum at $x = -2$ and local maximum value is obtained from (1) by substituting the value $x = -2$

$f(-2) = 2(-2)^3+3(-2)^2-12(-2)$

$= -16+12+24$

$= 20.$

When

x = 1

$f''(1) = 12(1) + 6$

$= 18.$

$f(x)$ attains local minimum at $x = 1$ and the local minimum value is obtained by substituting $x = 1$ in (1).

$f(1) = 2(1) + 3(1) - 12(1)$

$= -7$

Extremum values are – 7 and 20.

Example 4.8.2

Find the absolute (global) maximum and absolute minimum of the function $f(x)=3x^5-25x^3+60x+1$ in the interval [–2,2]

Solution :

$f(x) = 3x^5-25x^3+60x+1 \ldots (1)$

$f'(x) = 15x^4-75x^2+60$

$= 15(x^4-5x^2+4)$

$f'(x) = 0 \Longrightarrow 15(x^4-5x^2+4)= 0$

$\Longrightarrow (x^2-4)(x^2-1)= 0$

$x = \pm 2$ (or) $x = \pm 1$

of these four points $-2, \pm 1 \in [-2, 1]$ and $2 \notin [-2,1]$

From (1)

$f(-2) = 3(-2)^5 - 25(-2)^3 + 60(-2) + 1$

$= -15$

When $x= 1$

$f(1) = 3(1)^5 - 25(1)^3 + 60(1) + 1$

$= 39$

When $x = -1$

$f(-1) = 3(-1)^5 - 25(-1)^3 + 60(-1) + 1$

$= -37.$

Absolute maximum is 39

Absolute minimum is -37

4.9 Euler's theorem on homogeneous functions

Consider the expression $(x, y) = a_0x_n + a_1x_{n-1}y + a_2x_{n-2}y_2 + \cdots + a_ny_n$ … (1)

The degree of each term in the above expression is 'n'. Such an expression is called a homogeneous function of degree 'n'. A function f(x,y) is said to be a homogeneous function in x and y of degree 'n' if $f(tx, ty) = t^n f(x, y)$.

Theorem 4.9.1

Let f(x,y) be a homogeneous function of order n so that

- Euler's theorem states that if $f(x_1, x_2, \ldots, x_n)$ is a homogeneous function of degree k, then: $\sum_{i=1}^{n} x_i \frac{\partial f}{\partial x_i} = kf(x_1, x_2, \ldots, x_n)$
- In other words, the sum of the products of each variable and its partial derivative with respect to that variable is equal to k times the original function.

Total derivatives

Definition 4.9.1

The total derivative of a function of multiple variables represents how the function changes with respect to changes in all of its variables, not just one at a time.

Suppose we have a function $f(x_1, x_2, \ldots, x_n)$ which depends on n variables. The total derivative of f with respect to a variable x_i (where i ranges from 1 to n) is denoted as $\frac{df}{dx_i}$ and is defined as:

$$\frac{df}{dx_i} = \frac{\partial f}{\partial x_1}\frac{dx_1}{dx_i} + \frac{\partial f}{\partial x_2}\frac{dx_2}{dx_i} + \cdots \ldots + \frac{\partial f}{\partial x_n}\frac{dx_n}{dx_i}$$

Here, $\frac{df}{dx_i}$ represents the total derivative of f with respect to x_i, and $\frac{dx_j}{dx_i}$ represents the rate of change of variable x_j with respect to x_i.

4.10 Lagrange's method of undetermined multipliers

To determine the minimum or maximum value of a function f(x) subject to the equality constraint g(x) = 0 will form the Lagrange`s function as:

$L(x, \lambda) = f(x) - \lambda g(x)$

Here, L = Lagrange function of the variable x, λ = Lagrange multiplier

Important Notes on Partial Derivatives:

- While finding the partial derivative with respect to a variable, all the other variables should be considered as constants.
- The order of taking derivatives doesn't matter in partial derivatives. i.e., $\partial^2 f/\partial x\partial y = \partial^2 f/\partial y\partial x$.
- The rules of derivatives apply for partial differentiation as well.
- Instead of using the limit definition, applying derivative formulas make the process of finding the partial derivatives easier.

4.11 Summary

A partial derivative of a function of multiple variables measures how the function changes with respect to changes in one variable while keeping all other variables constant. Partial derivatives are denoted using the ∂ symbol, which represents a partial differentiation. In geometric terms, the partial derivative represents the slope of the tangent line to the surface defined by the function in the direction of the specified variable. Understanding partial derivatives is fundamental for analyzing functions of multiple variables and solving problems in various fields involving interdependent quantities. They provide a powerful tool for studying the behavior of functions in complex systems and optimizing their performance.

4.12 Keywords

- Partial Derivative
- Higher order partial derivative
- Rules of Partial derivation
- Maxima and Minima
- Lagrange's method

4.13 Self – Assessment Questions

1. Find $\frac{\partial f}{\partial x}$ for $f(x,y) = 3x^2y + 2xy^3$.
2. Compute $\frac{\partial g}{\partial y}$ for $g(x,y) = e^{xy} + \ln(y)$.
3. Find $\frac{\partial h}{\partial x}$ for $h(x,y) = \sin(xy) + x\cos(y)$.
4. Compute $\frac{\partial f}{\partial y}$ for $f(x,y) = \frac{x^2}{y} + \sqrt{y}$.
5. Find $\frac{\partial g}{\partial x}$ for $g(x,y) = \frac{1}{x} + y^2$.
6. Compute $\frac{\partial h}{\partial y}$ for $h(x,y) = e^{xy} \cdot \cos(x)$.
7. Find $\frac{\partial f}{\partial x}$ for $f(x,y) = \ln(x) + e^{2y}$.
8. Compute $\frac{\partial g}{\partial y}$ for $g(x,y) = \frac{x}{y} + \sin(xy)$.
9. Find $\frac{\partial h}{\partial x}$ for $h(x,y) = \frac{\sqrt{x}}{y} + \frac{1}{x^2}$.
10. Compute $\frac{\partial f}{\partial y}$ for $f(x,y) = x^3y^2 + \frac{1}{y}$.

4.14 Case Study

a) A metal sheet measuring twelve meters by eight meters is to be used to build a rectangular box with an open top. Determine the box's dimensions that will require the least quantity of metal.

b) How may partial differentiation be used to address the optimization issue mentioned in section (a)?

4.15 References

1. Arfken, G. B., & Weber, H. J. (2005). Mathematical Methods for Physicists (6th ed.). Academic Press.
2. Strang, G. (2005). Introduction to Linear Algebra (4th ed.). Wellesley-Cambridge Press.

CHAPTER - 5

Lagrange's and Cauchy's Forms

Learning Objectives:

- Recognize the two forms of Taylor's theorem: Lagrange's form and Cauchy's form.
- Understand the implications of Lagrange's form in approximating functions using Taylor polynomials.
- Apply Cauchy's form to approximate functions and analyze the error in approximation.

Structure:

5.1 Cauchy's Mean Value Theorem

Cauchy's mean value theorem is a generalization of the normal mean value theorem. This theorem is also known as the Extended or Second Mean Value Theorem. The normal mean value theorem describes that if a function f (x) is continuous in a close interval [a, b] where (a≤x ≤b) and differentiable in the open interval [a, b] where (a < x< b), then there is at least one point x = c on this interval, given as

f(b) - f (a) = f' (c) (b-a)

It establishes the relationship between the derivatives of two functions and changes in these functions on a finite interval.

Let's consider the function f(x) and g(x) be continuous on an interval [a, b] differentiable on (a,b), and g'(x) is not equal to 0 for all $x\epsilon(a,b)$. Then there is a point x = c in this interval given as

$$\frac{f(b)-f(a)}{g(b)-g(a)}=\frac{f'(c)}{g'(c)}$$

Proof of Cauchy's mean value theorem

$$\frac{f(b)-f(a)}{g(b)-g(a)}=\frac{f^{'}(c)}{g^{'}(c)}$$

Here, the denominator in the left side of the Cauchy formula is not zero: $g(b)-g(a) \neq 0$.

If $g(b) = g(a)$, then by Rolle's theorem, there is a point $d\epsilon(a,b)$, in which $g'(d) = 0$. Therefore, contradicts the hypothesis that $g'(x) \neq 0$ for all $x\epsilon(a,b)$.

Now, we apply the auxiliary function.

$F(x) = f(x) + \lambda g(x)$

And select λ in such a way to satisfy the given condition

$F(a) = f(b)$. we get,

$f(a) + \lambda g(a) = f(b) + \lambda g(b)$

$f(b)-f(a) = \lambda[g(a)- g(b)]$

$$\lambda=\frac{f(b)-f(a)}{g(a)- g(b)}$$

And the function F (x) exists in the form

$$F(x)=f(x)\frac{f(b)-f(a)}{g(b)- g(a)}g(x)$$

The function F (x) is continuous in the closed interval (a≤x ≤b), differentiable in the open interval (a < x< b) and takes equal vales at the endpoints of the interval. So, it satisfies all the conditions of Rolle's theorem. Then, there is a point c exist in the interval (a,b) given as

$F'(c) = 0$.

It follows that

$$f^{'}(c)=\frac{f(b)-f(a)}{b- a}g^{'}(c)$$

or

$$\frac{f(b)-f(a)}{g(b)-g(a)}=\frac{f^{'}(c)}{g^{'}(c)}$$

By putting g (x) = x in the given formula, we get the Lagrange formula:

$$f^{'}(c)=\frac{f(b)-f(a)}{b- a}$$

Cauchy's mean value theorem has the given geometric meaning. Consider the parametric equations give a curve $X = f(t)$ and $Y = g(t)$, where the parameter t lies in interval [a,b].

When we change the parameter t, the point of the curve in the given figure runs from $A(f(a).g(a))$ to $B(f(b).g(b))$.

According to Cauchy's mean value theorem, there is a point (f(c), g(c)) on the curve, where the tangent is parallel to the chord linking of two ends A and B of the curve.

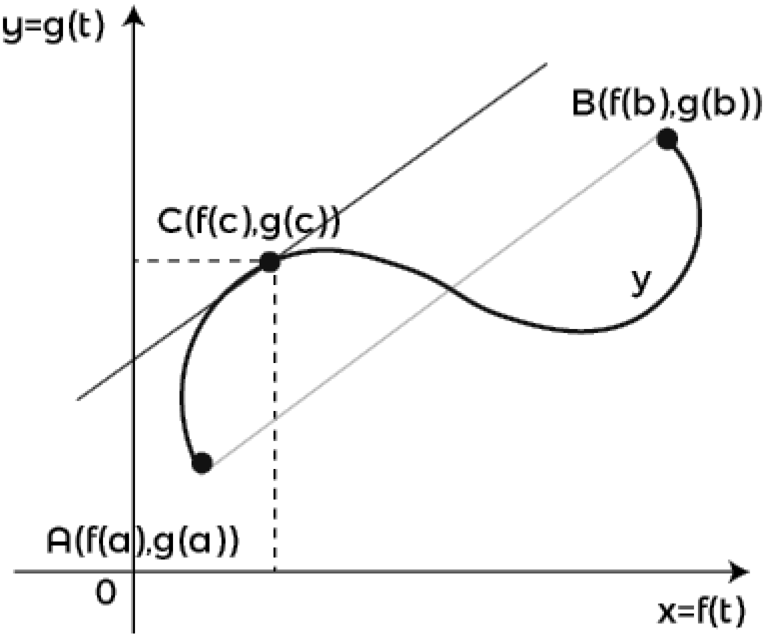


Figure 5.1.1 Cauchy function

Example 5.1.1

Calculate the value of xfor the following function, which satisfies the Mean Value Theorem,

F(x) = x^2 + 2x + 2

Solution:

Given

f(x) = x^2 + 2x + 2

According to Mean Value theorem,

$$f^{'}(c) = \frac{f(b) - f\ (a)}{b - \ a}$$

$$f^{'}(c) = 2c + 2$$

$$2c + 2 = \frac{(1 + 2 + 1) - \ (4 + 4 + 1)}{2 - \ 1} = -5$$

2c = -7

c = -7/2

Example 5.1.2

Calculate the value of x for the following function, which satisfies the Mean Value Theorem, F(x) = $x^2 + 4x + 7$

Solution:

Given

f(x) = $x^2 + 4x + 7$

According to Mean Value theorem,

$$f^{'}(c) = \frac{f(b) - f\ (a)}{b -\ a}$$

$$f^{'}(c) = 2c + 4$$

$$2c + 4 = \frac{(1 + 2 + 1) -\ (4 + 4 + 1)}{4 -\ 1} = -7$$

2c = -11

C = -11/2

5.2 Lagrange's Mean Value Theorem

Lagrange's mean value theorem is also known as the mean value theorem or MVT or LMVT. It states that if a function f (x) is a continuous in a close interval [a, b] where (a≤x ≤b) and differentiable in open interval [a, b] where (a < x< b), then there is at least one point x = c on this interval, given as

$$\boldsymbol{f(b)\ -\ f\ (a)\ =\ f'\ (c)\ (b - a)}$$

The above theorem is called first mean value theorem. It enables to express the increment of a function in the given interval through the derivative value at an intermediate point of the segment.

Proof of Lagrange's mean value theorem

Consider the auxiliary function given below

F (x) = f (x) + x

Here, we will select a number so that the condition $F\ (a)\ =\ F\ (b)$ is satisfied. Then

$$f(a) + \lambda a = f(b) + \lambda b$$

$$= f(a) - \lambda(a + b)$$

$$\lambda = -\frac{f(b) - f\ (a)}{b -\ a}$$

Then, the result, we get

$$F(x) = f(x)\frac{f(b) - f\ (a)}{b - a}x$$

The function F (x) is continuous in the closed interval ($a \le x \le b$), differentiable in the open interval ($a < x < b$), and takes equal values at the endpoints of the interval. So, it satisfies all the conditions of Rolle's theorem. Then, a point c exist in the interval (a,b) given as

F' (c) = 0.

It follows that

$$f'(c) - \frac{f(b) - f(a)}{b - a} = 0$$

5.3 Summary

- Cauchy's Mean Value Theorem, named after Augustin-Louis Cauchy, states that if two functions f(x) and g(x) are continuous on the closed interval [a,b] and differentiable on the open interval (a, b), then there exists a point c in (a, b) such that:

  $$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

- Lagrange's Mean Value Theorem, named after Joseph-Louis Lagrange, is a special case of Cauchy's Mean Value Theorem where the two functions f(x) and g(x) are the same function.
- It states that if a function f(x) is continuous on the closed interval [a,b] and differentiable on the open interval (a,b), then there exists a point c in (a,b) such that:

  $$\boldsymbol{f(b) - f(a) = f'(c)(b - a)}$$

  Both theorems are fundamental in calculus and play significant roles in proving other theorems and solving various problems related to rates of change and functions.

5.4 Keywords

- Cauchy's mean value theorem
- Lagrange's mean value theorem

5.5 Self – Assessment Questions

1. What does Cauchy's Mean Value Theorem state?
2. Describe the conditions under which Cauchy's Mean Value Theorem can be applied.
3. Provide an example of how Cauchy's Mean Value Theorem can be used to solve a problem involving two functions.
4. Discuss a real-world scenario where Cauchy's Mean Value Theorem could be applied.
5. State Lagrange's Mean Value Theorem.

6. Compare and contrast Lagrange's Mean Value Theorem with Cauchy's Mean Value Theorem.
7. Explain the conditions under which Lagrange's Mean Value Theorem holds.
8. How does Lagrange's Mean Value Theorem guarantee the existence of a point with a certain property?

5.6 Case Study

a) Describe how the estimation of the error of Taylor series approximations differs between Lagrange's form and Cauchy's form.
b) Talk about the circumstances in which estimating the error in Taylor series approximations can be done using Lagrange's form and Cauchy's form.
c) Give an example of a function and expansion point for which the error estimates produced by Lagrange's form and Cauchy's form differ noticeably.

5.7 References

1. Apostol, T. M. (1967). Calculus, Volume 1: One-Variable Calculus, with an Introduction to Linear Algebra (2nd ed.). John Wiley & Sons.
2. Courant, R., & John, F. (1999). Introduction to Calculus and Analysis, Vol. 1 (2nd ed.). Springer.

CHAPTER - 6

Maclaurin's series of sinx , cosx , e^x, log(l+x), $(l+x)^m$

Learning Objectives:

- Understand the concept of approximating functions using Maclaurin series
- Apply the derived Maclaurin series to approximate values of the function.
- Apply Maclaurin series to approximate functions and evaluate their values at specific points.

Structure:

6.1 Rolle's Theorem and The Mean Value Theorem

The Mean Value Theorem is most significant theorems in calculus.

Rolle's theorem states that if the output so differentiable function *f* are equal at the end points of an interval, then there must be an interior point *c* where $f'(c)$=0. Figure 6.1.1 illustrates this theorem.

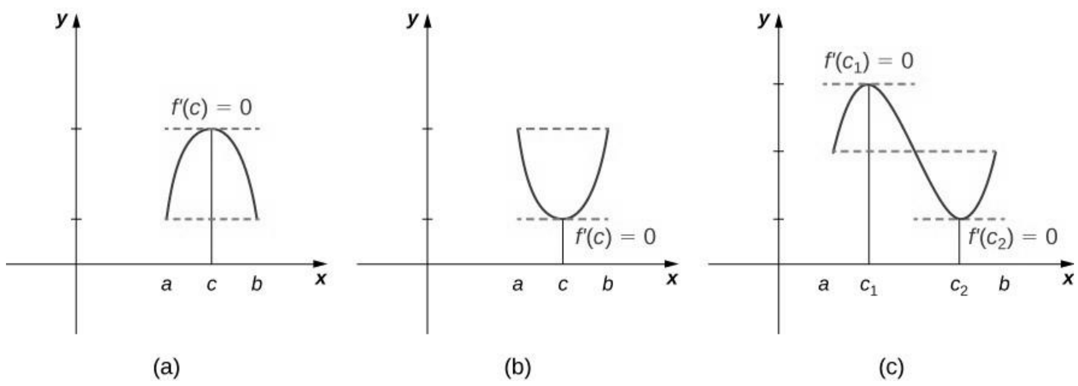


Figure 6.1.1: If a differentiable function *f* satisfies *f*(*a*) = *f*(*b*), then its derivative must be zero at some point(s) between *a* and *b*.

Theorem 6.1.1: Rolle's theorem statement

Let f be a continuous function over the closed interval $[a,b]$ and differentiable over the open interval (a,b) such that $f(a) = f(b)$. There then exists at least one $c \in (a,b)$ such that $f'(c)=0$.

Proof

Let $k = f(a) = f(b)$.

We consider three cases:

1. $f(x)=k$ for all $x\in(a,b)$.
2. There exists $x\in(a,b)$ such that $f(x)>k$.
3. There exists $x\in(a,b)$ such that $f(x)<k$.

Case 1: If$f(x)=k$ for all $x\in(a,b)$, then $f'(x) = 0$ for all $x\in(a,b)$.

Case 2: Since f is a continuous function over the closed, bounded interval $[a,b]$, by the extreme value theorem, it has an absolute maximum. Also, since there is a point $x\in(a,b)$ such that $f(x)>k$, the absolute maximum is greater than k. $f'(c)=0$.

Case 3: The case when there exists a point $x\in(a,b)$such that $f(x)<k$ is analogous to case2, with maximum replaced by minimum.

Rolle`s Theorem is a special case of the Mean Value Theorem. In Rolle's theorem, we consider differentiable functions f that are zero at the endpoints. The Mean Value Theorem generalizes Rolle's theorem by considering functions that are not necessarily zero at the endpoints.

Theorem 6.1.2: Mean Value Theorem

If a function f(x) is continuous on the closed interval [a, b] and differentiable on the open interval (a, b), then there exists at least one point c in the open interval (a, b) such that the instantaneous rate of change of f at c is equal to the average rate of change of f over the interval [a, b]. Mathematically, this can be expressed as:

$f'(c) - \frac{f(b)-f(a)}{b-a}$.

Proof:

The proof follows from Rolle's Theorem by introducing an appropriate function that satisfies the criteria of Rolle's Theorem. Consider the line connecting $(a, f(a))$ and $(b, f(b))$. Since the slope of that line is

$$\frac{f(b) - f(a)}{b - a}$$

and the line passes through the point $(a, f(a))$ the equation of that line can be written as

$$y = \frac{f(b) - f\ (a)}{b -\ a}(x - a) + f(a)$$

Let g(x) denote the vertical difference between the point (r.f(z)) and the point (r, y) on that line. Therefore,

$$g(x) = f(x) - \left[\frac{f(b) - f\ (a)}{b -\ a}(x - a) + f(a)\right]$$

Since f is a differentiable function over (a, b), g is also a differentiable function over (a, b). Furthermore, since f is continuous over [a, b], g is also continuous over [a, b]. Therefore. g satisfies the criteria of Rolle's theorem. Consequently, there exists a point $c \in (\mathrm{a}, \mathrm{b})$ such that $\mathrm{g}^{'}(\mathrm{c}) = 0$. Since

we see that

$$g'(x) = f'(x) - \frac{f(b) - f\ (a)}{b - a},$$

since $\mathrm{g}^{'}(\mathrm{c}) = 0$, we conclude that

$$f'(c) - \frac{f(b) - f\ (a)}{b - a}.$$

6.2 The Taylor Series

Let's assume we are given a function, *f*(*x*), and we will see if there exists a power series representation for that function.

We begin by **assuming** that we can let *f*(*x*) be represented by a power series:

$$f(x) = \sum_{k=0}^{\infty} a_k (x - c)^k = a_0 + a_1(x - c) + a_2(x - c)^2 + a_3(x - c)^3 + \ldots$$

But, we do not have the coefficients a_0, a_1, a_2, ... and need to find a way to calculate them.

As a first step, substituting $x = c$ into our power series yields $a_0 = f(c)$. To get the next coefficient, a_1, assuming we are able to differentiate our given function, we would obtain

$$f'(x) = 0 + a_1 + 2a_2(x - c) + 3a_3(x - c)^2 + \ldots$$

Substituting $x = c$ into this expression gives us an expression for a_1,

$$a_1 = \frac{df}{dx}(c)$$

So far, we have two of the coefficients, a_0 and a_1. Continuing with this process, provided that we are able to differentiate our given function again, we would obtain:

$$f''(x) = 0 + 2a_2 + +6a_3(x - c) + \ldots$$

Again, substituting $x = c$ into this expression yields

$$a_2 = \frac{1}{2}\frac{d^2 f}{dx^2}(c)$$

Continuing with this process, we can find as many coefficients as we need, provided that we can differentiate our given function as many times as needed. In other words, if we can differentiate our function N times, we can calculate N power series coefficients, and calculate the series expansion coefficients up to N terms. If we can differentiate our function an **infinite** number of times, we can find a power series expansion with an infinite number of terms.

The specific power series representation we introduced here derived what is known as the **Taylor Series** expansion. We will later discuss the conditions that are necessary for a function to have a Taylor series.

We have now found a method to calculate power series expansions for functions who have a certain property: that they have derivatives at all orders. Functions that have derivatives at all orders, on some open interval, are referred to as **analytic** on that interval. Functions that are analytic on an interval have what is called a **Taylor series expansion**.

Definition 6.2.1

Suppose that a given function, $f(x)$, is analytic on an **open** interval that contains the point $x = c$. The **Taylor series expansion for $f(x)$ at c** is

$$\sum_{k=0}^{\infty} a_k (x-c)^k,$$

and the coefficients of the series, a_k are given by

$$a_k = \frac{f^{(k)}(c)}{k!}$$

Here we are using the notation $f^{(k)}$ to denote the k^{th} derivative.

The Taylor series obtained when we let $c = 0$ is referred to a **Maclaurin series**.

When a Function Equals its Taylor Series

It is possible to show that if a given function is analytic on some interval, then it is *equal* to its Taylor series on that interval. That is, on an interval where $f(x)$ is analytic,

$$f(x) = \sum_{k=0}^{\infty} a_k (x-c)^k = \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k$$

We will not prove this result here, but the proof can be found in most first year calculus texts. The proof involves

- a derivation for an expression for the difference, or remainder, between *f* and the N^{th} order partial sum of a power series expansion, and
- shows that if and only if the remainder goes to zero when *N* goes to infinity, the Taylor series converges to *f*(*x*).

There are functions that are not equal to its Taylor series expansion. But for the purposes of this module, we will assume that all functions can be expanded as a Taylor series.

6.3 Maclaurin Expansion of e^x

Example 6.3.1

Find the Taylor series expansion for *ex* when x is zero, and determine its radius of convergence.

Solution:

Before starting this problem, note that the Taylor series expansion of any function about the point $c = 0$ is the same as finding its Maclaurin series expansion.

Step 1: Find Coefficients

Let $f(x) = e^x$. To find the Maclaurin series coefficients, we must evaluate

$$\left(\frac{d^k}{dx^k}f(x)\right)\Bigg|_{x=0}$$

for $k = 0, 1, 2, 3, 4, ...$

Because $f(x) = e^x$, then all derivatives of $f(x)$ at $x = 0$ are equal to 1. Therefore, all coefficients of the series are equal to 1.

Step 2: Substitute Coefficients into Expansion

By substitution, the Maclaurin series for e^x is

$$\sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!}(x-0)^k = \frac{x^0}{0!}+\frac{x^1}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\frac{x^5}{5!}+\ldots = \sum_{k=0}^{\infty}\frac{x^k}{k!}$$

Step 3: Radius of Convergence

The ratio test gives us:

$$\lim_{k\to\infty}\left|\frac{x^{k+1}}{(k+1)!}\Big/\frac{x^k}{k!}\right| = \lim_{k\to\infty}\frac{|x|}{k+1} = 0$$

Because this limit is zero for all real values of *x*, the radius of convergence of the expansion is the set of all real numbers.

Explanation of Each Step

Step 1:

Maclaurin series coefficients, a_k are always calculated using the formula

$$a_k = \frac{f^{(k)}(0)}{k!}$$

Where f is the given function, and in this case is e^x. In step 1, we are only using this formula to calculate coefficients. We found that all of them have the same value, and that value is one.

Step 2:

Step 2 was a simple substitution of our coefficients into the expression of the Taylor series given on the previous page.

Step 3:

This step was nothing more than substitution of our formula into the formula for the ratio test. Because we found that the series converges for all *x*, we did not need to test the endpoints of our interval. If however we did find that the series only converged on an interval with a finite width, then we may need to take extra steps to determine the convergence at the boundary points of the interval.

Write the Maclaurin series expansion of the following functions: (i) e^x (ii) sinx (iii) cosx (iv) log (1 – x); -1 ≤ x < 1 (v) tan-1 (x) ; -1 ≤ x ≤ 1

(i) e^x

$$f(x) = e^x \ ; \quad f(0) = e^0 = 1$$
$$f'(x) = e^x \ ; \quad f'(0) = 1$$
$$f''(x) = e^x \ ; \quad f''(0) = 1$$
$$f(x) = e^x = 1 + \frac{1 \cdot x}{1!} + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 \ldots$$
$$= 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} \ldots \text{ holds for all } x$$

(ii) sin x

$f(x) = \sin x \quad ; \quad f(0) = 0$
$f'(x) = \cos x \quad ; \quad f'(0) = 1$
$f''(x) = -\sin x \quad ; \quad f''(0) = 0$
$f'''(x) = -\cos x \quad ; \quad f'''(0) = -1$
$f^4(x) = \sin x \quad ; \quad f^4(0) = 0$
$f^5(x) = \cos x \quad ; \quad f^5(0) = 1$

The Maclaurin expansion of $f(x)$ is

$$f(x) = f(0) + \frac{x^1}{\lfloor 1} f'(0) + \frac{x^2}{\lfloor 2} + f''(0) + \ldots$$

$$f(x) = \sin x = \frac{x^1}{\lfloor 1}(1) + \frac{x^3}{\lfloor 3}(-1) + \frac{x^5}{\lfloor 5}(1) + \ldots$$

$$= x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} - \ldots$$

(iii) cos x

$f(x) = \cos x \quad ; \quad f(0) = 1$
$f'(x) = -\sin x \quad ; \quad f'(0) = 0$
$f''(x) = -\cos x \quad ; \quad f''(0) = -1$
$f^3(x) = \sin x \quad ; \quad f^3(0) = 0$
$f^4(x) = \cos x \quad ; \quad f^4(0) = 1$

The Maclaurin's expansion

$$f(x) = f(0) + \frac{x^1}{\lfloor 1} f'(0) + \frac{x^2}{\lfloor 2} f''(0) + \ldots$$

$$f(x) = \cos x = 1 + \frac{x^2}{\lfloor 2}(-1) + \frac{x^4}{\lfloor 4}(1) - \ldots$$

$$= 1 - \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} - \ldots$$

(iv) log (1 – x); -1 ≤ x < 1

$f(x) = \log(1-x)$

$f(0) = 0$

$f'(x) = \dfrac{1}{1-x}(-1) = \dfrac{-1}{1-x}$

$f'(0) = -1$

$f''(x) = -\dfrac{-1}{(1-x)^2}\ (-1) = \dfrac{-1}{(1-x)^2}$

$f''(0) = -1$

$f'''(x) = -\left(\dfrac{-2}{(1-x)^3}(-1)\right) = \dfrac{-2}{(1-x)^3}$

$f'''(0) = -2$

The Maclaurin's expansion of $f(x)$ is $f(x)$

$= f(0) + \dfrac{x^1}{\lfloor 1} f'(0) + \dfrac{x^2}{\lfloor 2} f''(0) + \ldots.$

Here $f(x) = \log(1-x)$

So $\log(1-x)$

$= 0 + \dfrac{x}{\lfloor 1}(-1) + \dfrac{x^2}{\lfloor 2}(-1) + \dfrac{x^4}{\lfloor 4}(-2)$

$= -x - \dfrac{x^2}{2} - \dfrac{x^3}{3} \ldots.$

6.4 Summary

1. **Maclaurin Series of $\sin(x)$:**
   $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}$

2. **Maclaurin Series of $\cos(x)$:**
   $\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}$

3. **Maclaurin Series of e^x:**
   $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

4. **Maclaurin Series of $\log(1+x)$:**
   $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n} \quad \text{for } -1 < x \leq 1$

5. **Maclaurin Series of $(1+x)^m$:**
   $(1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)}{3!}x^3 + \ldots = \sum_{n=0}^{\infty}\binom{m}{n}x^n \quad \text{for } -1 < x < 1$

6.5 Keywords

- Rolle’s Theorem
- Taylor's theorem

- Taylor's Series
- Maclaurin’s series

6.6 Self – Assessment Questions

1. Who is Maclaurin and what is his series known for?
2. What is a Maclaurin series?
3. How does a Maclaurin series differ from a Taylor series?
4. What is the general form of a Maclaurin Maclaurin series?
5. What are some common examples of functions with known Maclaurin series expansions?
6. How does the number of terms in a Maclaurin series affect its accuracy?
7. How do you find the Maclaurin series expansion of a function?
8. What is the significance of the Maclaurin series in calculus?
9. Can any function be represented by its. What is the relationship between the Maclaurin series and the derivatives of a function at a point?

6.7 Case Study

The function ln(1+x) is to be approximated by the first three terms of its Maclaurin series. i.e.,

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3}$$

Estimate the maximum value of x for which the approximation agrees with the exact value to 3 decimal places.

6.8 References

1. Stewart, J. (2016). Calculus: Early Transcendentals (8th ed.). Cengage Learning.
2. Larson, R., & Edwards, B. (2017). Calculus of a Single Variable (11th ed.). Cengage Learning.

CHAPTER -7

L-Hospital's Rule

Learning Objectives:

- Recognize when L-Hospital's Rule can be applied to evaluate limits of indeterminate forms.
- Identify situations where applying L-Hospital's Rule simplifies limit calculations.
- Use L-Hospital's Rule to analyze the behaviour of functions at points of discontinuity or singularity.

Structure:

7.1 L- Hospital's Rule:-

L-Hospital's rule (L-Hospital's rule) is a very important rule in calculus that is used to evaluate the limits that result in indeterminate forms (such as 0/0, ∞/∞, etc). These types of limits

- Can't be calculated by direct substitution of the limit and/or
- Can be evaluated but using a very long procedure.

Let us see the L-Hospital's Rule along with its statement, proof, and examples. Also, let us see some common misconceptions that may happen while the application of this rule.

History

L-Hospital's rule has various names such as L-Hospital's rule, L-Hospital's rule, Bernoulli's rule, etc, and is used to evaluate the limits of ind determinant forms. It was first introduced by a Swiss mathematician Johann Bernoulli in 1694 and hence it is known as Bernoulli's rule. It was later developed by a French mathematician Guillaume de L-Hospital's and hence it became popular with the name L-Hospital's rule.

Let us examine an example before going to see what this rule says: $\lim_{x \to 2} (x^2 - 4) / (x - 2)$. The first thing we usually do to evaluate a limit is substituting the limit. Let us see what happens when we apply the limit. We get $(2^2 - 4) / (2 - 2) = 0 / 0$. This is an indeterminate form (as 0 / 0 is not defined). But let us try to simplify this limit in a different way. We know that $x^2 - 4$ can be written as $(x + 2)(x - 2)$ (using$a^2 - b^2$). So $\lim_{x \to 2} (x^2 - 4) / (x - 2) = \lim_{x \to 2} [(x + 2)(x - 2)] / (x - 2) = \lim_{x \to 2} (x + 2) = 2 + 2 = 4$. Because of the factorization, we could evaluate this limit easily.

Let us take another limit $\lim_{x \to 0} (\sin x) / x$ (whose actual value is 1 and this will be proved later) which also gives us an indeterminate form 0 / 0 but this cannot be evaluated easily as the other one. L-Hospital's Rule simplifies the process of finding limits that give indeterminate forms by the application of limit. Before going to learn it, let us just see what can be the other indeterminate forms.

Indeterminate Forms:

The indeterminate form is something that cannot be defined mathematically. Indeterminate forms can be of the form $0/0$, $\pm\infty/\pm\infty$, $0 \times \infty$, $\infty - \infty$, 0^0, 1^∞, ∞^0, etc. But the most common indeterminate forms that encounter while the application of L-Hospital's rule are $0/0$ and $\pm\infty/\pm\infty$.

7.2 What is L-Hospital’s rule?

L-Hospital’s Rule is a technique used to evaluate limits of indeterminate forms, particularly those involving fractions where both the numerator and denominator approach zero (or infinity) as the independent variable approaches a certain value. It's named after the French mathematician Guillaume de L-Hospital’s

If $\lim_{x \to a} \frac{f(x)}{g(x)} = \text{indeterminate form}$

then :

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

.

L-Hospital’s Rule Formula

L-Hospital’s rule states that for any two continuous functions f(x) and g(x), $\lim_{x \to a}$ f(x)/g(x) is an indeterminate form, then $\lim_{x \to a} f(x) / g(x) = \lim_{x \to a} f'(x) / g'(x)$, where

- 'a' is any real number, or ∞, or - ∞.
- $\lim_{x \to a} f(x) / g(x)$ is an indeterminate form when x = a is applied.
- f'(x) is the derivative of f(x)
- g'(x) is the derivative of g(x) and $g'(a) \neq 0$.

7.3 When and How to Apply L-Hospital’s Rule?

L-Hospital’s rule should be applied only when $\lim_{x \to a} f(x) / g(x)$ leads to an indeterminate form by the direct application of limit. In such cases, we just differentiate the numerator and denominator (using derivative formula) separately and then apply the limit. Here are the same examples that were mentioned in the first section and are solved using L-Hospital’s rule very easily.

- $\lim_{x \to 2} (x^2 - 4) / (x - 2) = 0/0$ by the direct application of limit (x = 2).
  Using L-Hospital’s rule:
  $\lim_{x \to 2} (x^2 - 4) / (x - 2) = \lim_{x \to 2} (2x) / (1) = 2(2) = 4$
  (this follows from power rule)
- $\lim_{x \to 0} (\sin x) / x = 0 / 0$ when x = 0 is applied.
  Using L-Hospital’s rule:
  $\lim_{x \to 0} (\sin x) / x = \lim_{x \to 0} (\cos x) / 1 = \cos 0 = 1$
  (this follows from the derivative of sinx formula)

Sometimes, the limit still results in an indeterminate form even after the application of L-Hospital's rule for one time. In that case, we can apply the same rule again and again as required.

Applying L-Hospital's Rule Multiple Times

We can apply L-Hospital's rule as many times as required. Each time we apply the rule, we compute the derivatives of numerator and denominator functions separately and then apply the limit. But before each application of this rule, just make sure that the current limit leads to an indeterminate form. Here is an example.

Evaluate the limits at infinity

(a) $\lim\limits_{x\to\infty} \frac{e^x}{x}$

Solution

Direct "substitution" gives $\frac{\infty}{\infty}$ so we can use L'Hopital's Rule to give

$$\lim_{x\to\infty} \frac{e^x}{x} \overset{H}{=} \lim_{x\to\infty} \frac{e^x}{1} = \infty$$

(b) $\lim\limits_{x\to\infty} x^2 e^{-x}$

Solution

Write the limit as

$$\lim_{x\to\infty} x^2 e^{-x} = \lim_{x\to\infty} \frac{x^2}{e^x}$$

Then direct "substitution" gives $\frac{\infty}{\infty}$ so we can use L'Hopital's Rule to give

$$\lim_{x\to\infty} x^2 e^{-x} \overset{H}{=} \lim_{x\to\infty} \frac{2x}{e^x} \overset{H}{=} \lim_{x\to\infty} \frac{2}{e^x} = 0$$

(c) $\lim\limits_{x\to\infty} \left(\sqrt{x^2+x+1} - x\right)$

Solution

Direct "substitution" gives the indeterminate form $\infty - \infty$. Write the limit as

$$\lim_{x\to\infty} \left(\sqrt{x^2+x+1} - x\right) = \lim_{x\to\infty} x\left(\sqrt{1+\frac{1}{x}+\frac{1}{x^2}} - 1\right) = \lim_{x\to\infty} \frac{\left(\sqrt{1+\frac{1}{x}+\frac{1}{x^2}} - 1\right)}{\frac{1}{x}}$$

Now direct "substitution" gives $\frac{\infty}{\infty}$ so we can use L'Hopital's Rule to give

$$\lim_{x\to\infty} \left(\sqrt{x^2+x+1} - x\right) \overset{H}{=} \lim_{x\to\infty} \frac{\frac{1}{2}\left(1+\frac{1}{x}+\frac{1}{x^2}\right)^{-1/2}\left(-\frac{1}{x^2}-\frac{2}{x^3}\right)}{-\frac{1}{x^2}}$$

$$= \lim_{x\to\infty} \frac{1+\frac{2}{x}}{2\left(1+\frac{1}{x}+\frac{1}{x^2}\right)^{1/2}} = \frac{1}{2}$$

When can't we apply L-Hospital's Rule?

Application of L-Hospital's rule when the limit does NOT give an indeterminate form throws the wrong result. For example:

$\lim_{x\to 2}(3x + 1) / (2x + 2) = (3(2) + 1) / (2(2) + 2) = 7/6$ (correct answer)

The limit didn't result in an indeterminate form and so we can't apply the L-Hospital's rule. Let us see what we get if we apply. We know that the derivatives of 3x + 1 and 2x + 2 are 3 and 2 respectively. So the above limit after the application of rule becomes:

$\lim_{x\to 2}(3/2) = 3/2$ (incorrect answer)

So do apply the limit first and make sure that an indeterminate form is resulted before applying the rule.

Simplify the Fraction Before Each Application:

When we apply L-Hospital's rule for multiple times, simplify the rational expression each time before applying the limit every time. Otherwise, we result in a wrong answer. Here is an example.

$\lim_{x\to 1} (x^3 - 1) / (x^2 - 1)$

$= \lim_{x\to 1} (3x^2/2x)$

$= \lim_{x\to 1} (3x / 2)$

$= 3(1)/2$

$= 3/2$ (correct answer)

But what happens if we don't simplify the fraction (in the third step) and try to apply L-Hospital's rule again?

$\lim_{x\to 1} (x^3 - 1) / (x^2 - 1)$

$= \lim_{x\to 1} (3x^2/2x)$

$= \lim_{x\to 1} (6x / 2)$

$= \lim_{x\to 1} 3x$

$= 3(1)$

$= 3$ (incorrect answer)

Thus, simplify everything into the lowest terms before the application of the rule.

7.4 Important Notes on L-Hospital's Rule:

- The limit of a fraction of two functions (that result is equal to the limit of the fraction of their derivatives.
- Do not apply L-Hospital's rule if the limit is not resulting in an indeterminate form.

- We can apply L-Hospital's rule as many times as required but before the application of each time, we should check whether the limit in that particular step is giving indeterminate form.
- When we are trying to apply L-Hospital's rule for the product f(x) · g(x), first, write it as fraction (i.e., either as f(x) / (1 / g(x)) or as g(x) / (1/ f(x))).

Example 1

Evaluate the limit $\lim_{x\to 3} \frac{x^2+x-12}{x^2-9}$

Solution

Since direct substitution gives $\frac{0}{0}$ we can use L'Hopital's Rule to give

$$\lim_{x\to 3} \frac{x^2+x-12}{x^2-9} \stackrel{H}{=} \lim_{x\to 3} \frac{2x+1}{2x} = \frac{7}{6}$$

Example 2

Evaluate the limit $\lim_{x\to 0} \frac{\sin 3x}{\tan 4x}$

Solution

Since direct substitution gives $\frac{0}{0}$ we can use L'Hopital's Rule to give

$$\lim_{x\to 0} \frac{\sin 3x}{\tan 4x} \stackrel{H}{=} \lim_{x\to 0} \frac{3\cos 3x}{4\sec^2 4x} = \frac{3}{4}$$

Example 3

Evaluate the limit $\lim_{x\to \frac{\pi}{2}} \left(x-\frac{\pi}{2}\right)\tan x$ using L'Hopital's Rule.

Solution

Write the limit as

$$\lim_{x\to \frac{\pi}{2}} \left(x-\frac{\pi}{2}\right)\tan x = \lim_{x\to \frac{\pi}{2}} \frac{x-\frac{\pi}{2}}{\cot x}$$

Then direct substitution gives $\frac{0}{0}$ so we can use L'Hopital's Rule to give

$$\lim_{x\to \frac{\pi}{2}} \left(x-\frac{\pi}{2}\right)\tan x \stackrel{H}{=} \lim_{x\to \frac{\pi}{2}} \frac{1}{(-\csc^2 x)} = -1$$

7.5 Summary

L'Hospital's Rule is a method for finding limits of indeterminate forms. It applies to limits that result in the forms 0/0 or ∞/∞. Indeterminate forms arise in calculus when the limit of an expression is not immediately clear due to conflicting tendencies of its components. Understanding these concepts is crucial for effectively solving complex limits and handling expressions that initially seem undefined.

7.6 Keywords

- L-Hospital's rule
- Indeterminate Forms

7.7 Self – Assessment Questions

1. Explain how the form 0^0 can be interpreted differently depending on the context of the functions involved.
2. Define what an indeterminate form is. List the seven common indeterminate forms.
3. Determine if the following limits are in indeterminate forms:
    - $\lim_{x \to 0} \sin(x)/x$
    - $\lim_{x \to \infty}(1+1/x)^x$
    - $\lim_{x \to 0} x\ln(x)$

7.8 Case Study

1. Consider the following limit problem that presents an indeterminate form:

$\lim_{x \to 0}(\sin(3x)-3x)/x^3$.

7.9 References

1. Anton, H., Bivens, I., & Davis, S. (2010). Calculus Early Transcendentals (9th ed.). John Wiley & Sons.
2. Larson, R., & Edwards, B. (2017). Calculus of a Single Variable (11th ed.). Cengage Learning.

CHAPTER - 8

Tangents and Normal

Learning Objectives:

- Differentiate between tangents and normal based on their definitions and geometric properties.
- Apply the point-slope form or slope-intercept form to derive equations of tangent and normal lines.
- Interpret the geometric meaning of tangent and normal lines in relation to the curve's shape and direction.

Structure:

8.1 Tangents and normal:-

The following Figure 8.1.1 shows the tangent and normal to a circle at a point on its circumference.

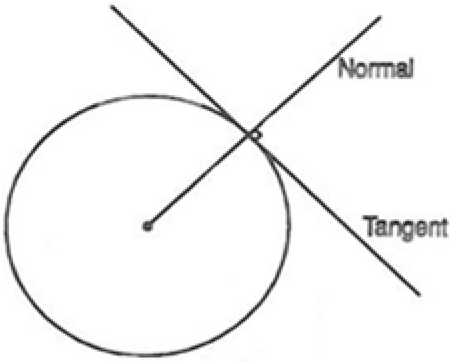


Figure 8.1.1 The tangent and normal to a circle at a point

Both the tangent and normal are straight lines of the tom y=mx+c and to find these lines we need two pieces of information, the gradient of the line and a point that the line passes through.

The normal to a circle will always pass through the centre of the circle.

The tangent and the normal are always at right angles to each other.

Example 8.1.1

Find the equation of the normal circle $x^2+y^2=9$ at the point $(1, \sqrt{8})$ on its circumference. By examining the equation of the circle you can see that the centre point is (0, 0).

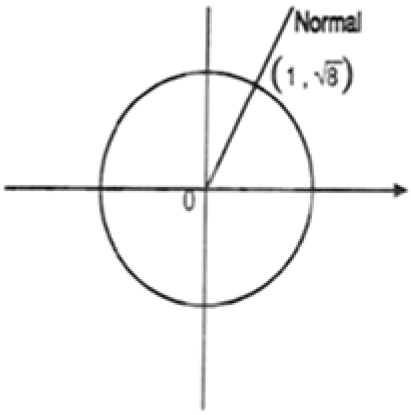


Figure 8.1.2 The tangent and normal to $x^2+y^2=9$

To find the gradients use $\frac{y_2-y_1}{x_2-x_1}$, where (x_1, y_1) and (x_2, y_2) are two points on the line

So the gradient = $\frac{\sqrt{8}-1}{1-0} = \sqrt{8}$ =M

So I've found one piece of information, now I need to use $y-y_1=M(x-x_1)$ to find the equation of the line.

I'll use the point (0, 0). I could use (1, $\sqrt{8}$) but I think (0, 0) would be easier!

So y-0 = $\sqrt{8}$ (x-0)

So y=$\sqrt{8}$x This is the equation of Normal.

So y-$\sqrt{8}$ =-(1/$\sqrt{8}$)(x- 1)

$\sqrt{8}y + x = 9$ Equation of the tangent.

8.2 Curvature

Curvature is a numerical measure of bending of the curve. At a particular point on the curve, a tangent can be drawn. Let the positive x-axis and this line form an angle Ψ. The magnitude of the rate of change of Ψ with respect to the arc length s is therefore defined as curvature.

∴Curvature at $P = \left|\frac{d\Psi}{ds}\right|$

It is obvious that smaller circle bends more sharply than larger circle and thus smaller circle has a larger curvature. Radius of curvature is the reciprocal of curvature and it is denoted by ρ.

8.3 Center of Curvature: The center of the circle that most closely resembles the curve at a given location is known as the center of curvature. Put another way, the center of curvature would be the place at which you would draw a circle that matched the curve's curvature and just touched it. From a mathematical perspective, it is the point where the distance along the normal line, which is perpendicular to the curve at that point, equals the radius of curvature. The concave side of the curve is where the center of curvature is located.

8.4 Chord of Curvature: The chord of curvature is a line segment connecting two points on the curve that are separated by a specified distance along the curve. This distance is typically the radius of curvature. The chord of curvature helps in approximating the curve by a straight line segment over a small interval, providing a linear approximation to the curve's behavior in that region.

- **Radius of curvature of Cartesian curve: $y = f(x)$**

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\left|\frac{d^2y}{dx^2}\right|} = \frac{(1+y_1^2)^{3/2}}{|y_2|} \text{ (When tangent is parallel to x – axis)}$$

$$\rho = \frac{\left[1+\left(\frac{dx}{dy}\right)^2\right]^{3/2}}{\left|\frac{d^2x}{dy^2}\right|} \text{ (When tangent is parallel to y – axis)}$$

- **Radius of curvature of parametric curve:**

  $x = f(t), y = g(t)$

$$\rho = \frac{\left(x'^2 + y'^2\right)^{3/2}}{|x'y'' - y'x''|}, \quad \text{where } x' = \frac{dx}{dt} \text{ and } y' = \frac{dy}{dt}$$

Example 8.4.1 Find the radius of curvature at any pt of the cycloid

$x = a(\theta + \text{Sin}\,\theta), \; y = a(1 - \cos\theta)$

Solution: $x' = \frac{dx}{d\theta} = a(1 + \cos\theta)$ and $y' = \frac{dy}{d\theta} = a\sin\theta$

$$x'' = \frac{d^2x}{d\theta^2} = -a\sin\theta \quad \text{and} \quad y'' = \frac{d^2y}{d\theta^2} = a\cos\theta$$

$$\text{Now } \rho = \frac{\left(x'^2 + y'^2\right)^{3/2}}{|x'y'' - y'x''|} = \frac{\{a^2(1+Cos\,\theta)^2 + a^2 sin^2\theta\}^{3/2}}{a^2(1+Cos\,\theta)\,Cos\,\theta + a^2\,sin^2\theta}$$

$$= \frac{a\left(1+cos^2\theta+2cos\theta+sin^2\theta\right)^{3/2}}{cos\theta+cos^2\theta+sin^2\theta}$$

$$= \frac{a(2+2cos\theta)^{3/2}}{1+cos\theta}$$

$$= 2\sqrt{2}\,a\sqrt{1+cos\theta}$$

$$= 2\sqrt{2}\,a\sqrt{2\frac{\cos^2\theta}{2}} = 4a\cos\frac{\theta}{2}$$

Radius of curvature of Polar curves $r = f(\Theta)$

$$\rho = \frac{\left(r^2 + r_1^2\right)^{3/2}}{2r_1^2 + r^2 - rr_2} \qquad \left(where\; r_1 = \frac{dr}{d\theta},\; r_2 = \frac{d^2r}{d\theta^2}\right)$$

Example 8.4.2 Prove that for the cardioide r = a (1 + cos θ),

$\frac{\rho^2}{r}$ is const.

Solution: Here r = a (1+ cos θ)

$$\Rightarrow r_1 = -\text{a Sin}\,\theta \text{ and } r_2 = -\text{a cos}\,\theta$$

$$\therefore r^2 + r_1^2 = a^2\,[(1+\cos\theta)^2 + \sin^2\theta] = 2a^2\,(1 + \cos\theta)$$

$$r^2 + 2r_1^2 - rr^2 = a^2[(1+\cos\theta)^2 + 2\sin^2\theta + \cos\theta(1+\cos\theta)]$$

$$= 3a^2\,(1+\cos\theta)$$

$$\therefore \rho^2 = \frac{(r^2+r_1^2)^3}{(r^2+2r_1^2-rr_2)^2} = \frac{8a^6(1+\cos\theta)^3}{9a^4\,(1+\cos\theta)^2} = \frac{8}{9}a^2\,(1+\cos\theta)$$

$$\Rightarrow \rho^2 = \frac{8a}{9}\text{r}$$

$\therefore \frac{\rho^2}{r} = \frac{8a}{9}$ which is a constant.

8.5 Radius of curvature at the origin by Newton's method

It is applicable only when the curve passes through the origin and has x-axis or y-axis as the tangent there. When x-axis is the tangent, then

$$\bar{\rho} = \lim_{x\to 0}\frac{x^2}{2y}$$

When y-axis is the tangent, then

$$\bar{\rho} = \lim_{x\to 0}\frac{y^2}{2x}$$

Example 8.5.1 Find the radius of curvature at the origin of the curve

$$x^3y - xy^3 + 2x^2y + xy - y^2 + 2x = 0$$

Solution: Tangent is x = 0 ie y–axis,

$$\rho = \lim_{y\to 0}\frac{y^2}{2x}$$

Dividing the given equation by 2x, we get

$$\frac{x^3y}{2x} - \frac{xy^3}{2x} + \frac{2x^2y}{2x} + \frac{+xy}{2x}\ \frac{-y^2}{2x} + \frac{2x}{2x} = 0$$

$$x^3\left(\frac{y}{2x}\right) - xy\left(\frac{y^2}{2x}\right) + xy + x\left(\frac{y}{2x}\right) - \left(\frac{y^2}{2x}\right) + 1 = 0$$

Taking limit $y \to 0$ on both the sides , we get $\rho = 1$

8.6 Summary

Tangent: A line that touches a curve at a single point without crossing through it. It represents the instantaneous direction of the curve at that point. • Normal: A line perpendicular to the tangent at a given point on a curve. Curvature measures how much a curve deviates from being a straight line at a particular point. It quantifies the rate at which the tangent direction changes as one moves along the curve. The center of curvature is the center of the circle that best approximates the curve at a given point. It lies on the normal to the curve at that point.

8.7 Keywords

- Tangents and Normal
- Radius of Curvature
- Polar Curve

8.8 Self – Assessment Questions

1. Define a tangent to a curve.
2. What is the slope of the tangent to a curve at a given point?
3. How is the equation of a tangent line to a curve at a given point derived?
4. What is the normal line to a curve?
5. How is the slope of the normal line related to the slope of the tangent line?
6. What is the condition for a line to be normal to a curve at a given point?
7. Can a curve have more than one tangent at a given point?
8. Can a curve have more than one normal at a given point?
9. How is the normal line equation derived at a point on a curve?
10. How are tangents and normal useful in calculus and real-world applications?

8.9 Case Study

Riya is studying in class XI. Today her brother Mayank is teaching her the topic of Maths 'Tangent and Normal'. Mayank prepared the following notes on 'Tangent and Normal' for Riya.

(i) Slope of gradient of a line: If a line makes an angle q is called the slope of gradient of the line.

(ii) Pictorial representation of tangent & normal

(iii) Facts about the slope of aline:

(a) If a line is parallel to the x-axis (or perpendicular to y-axis), then its slope is 0 (Zero).

(b) If a line is parallel to the y-axis (or perpendicular x-axis), then its slope is $\frac{1}{0}$i.e., not defined.

(c) If two lines are perpendicular, then product of their slope equals -1 $i.e., m1 \times m2 = -1$. Whereas, for two parallel lines, their slopes are equal $i.e., m1 = m2$. (Here in both the cases, $m1$ and $m2$ represent the slope of the respective line.

8.10 References

1. Stewart, J. (2016). Calculus: Early Transcendentals (8th ed.). Cengage Learning.
2. Larson, R., & Edwards, B. (2017). Calculus of a Single Variable (11th ed.). Cengage Learning.

CHAPTER - 9

Tracing of Curves

Learning Objectives:

- Recognize the goal of curve tracing, which is to analyze and understand the behaviour of curves represented by functions or parametric equations.
- Learn to identify and analyze key features of curves, such as intercepts, asymptotes, critical points, and inflection points.
- Apply curve tracing techniques to sketch graphs of functions and parametric curves.

Structure:

9.1 Tracing of curves

We can sketch any function using Curve Sketching but some functions can be quite tricky to sketch. Let's consider some examples of various different functions which we will sketch using the techniques of curve sketching.

9.2 Linear Functions

Sketching Linear Function is quite an easy task in curve sketching as we just need two points on the graph and the line joining those two points is the graph of a linear function. Let's consider an example:

Example 9.2.1

Draw the graph for the function, $\mathcal{F}$ (x) = 2x + 3.

Solution:

For $\mathcal{F}$ (x) = 2x + 3,

Put x = 0 ⇒ $\mathcal{F}$ (0) = 3

and Put x = 1 ⇒ $\mathcal{F}$ (1) = 5

Now, draw a striaght line passing through the pionts (0, 3) and (1, 5) which is the chart of the linear function $\mathcal{F}$(x) = 2x+3.

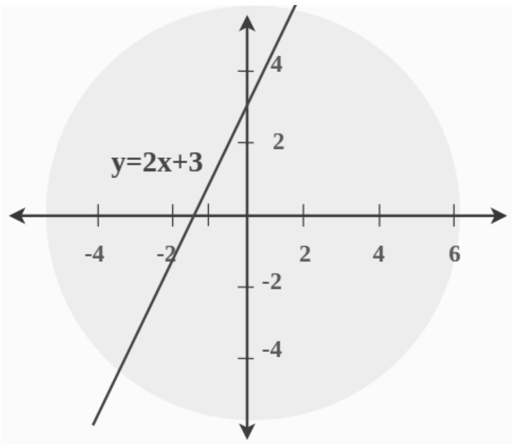


Figure 9.2.1 Striaght line passing through the pionts (0, 3) and (1, 5) on $\mathcal{F}$(x) = 2x+3.

9.3 Polynomial Functions

Polynomial functions occur a lot in calculus, and it is essential to know how to sketch their graphs. We will look at a function and use the techniques studied above to infer the graph of the function. The general idea is to look for asymptomatic values, and where they are going, and then find the critical points and draw a graph according to them. Let's see it through examples,

Example 9.3.1

Draw the graph for the given function,

$\mathcal{F}$ (x) = $x^2 + 4$

Solution:

We know that the domain of this function is all real numbers. This function will tend to infinity as we go towards large positive and negative values of x.

Notice that $\mathcal{F}$ (-x) = $(-x)^2 + 4 = x^2 + 4$ = $\mathcal{F}$ (x). That is this function is even, so its graph must be symmetric about the y-axis.

Now we know that graph goes to infinity and is symmetrical around the y-axis. Now, let's look for critical points.

$\mathcal{F}$(x) = 2x = 0

⇒ x = 0

Thus, there is only one critical point which is x = 0. Checking the double derivative $\mathcal{F}$''(x) = 2. Since $\mathcal{F}$''(x) > 0 for every x. So, the graph must be convex upward everywhere with minima at x = 0. Now we just need to know the value of the function at minima.

$\mathcal{F}$ (0) = 4.

Now we are ready to plot a graph.

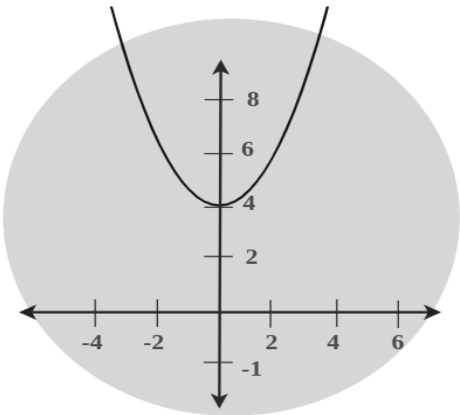


Figure 9.3.1 Minima of $\mathcal{F}$ (x) = $x^2 + 4$

.

Exponential Functions

Exponential functions are an essential part of calculus and are commonly represented as $\mathcal{F}$ (x) = a^x, where a is any positive constant and x can be any possible real number. To sketch the graph of Exponential Functions we need to check, the domain, range, and asymptotes. We also need to check whether the function is increasing or decreasing. If the base of the exponential function lies between 0 and 1 then it decreases in its domain otherwise it is an increasing function.

Let's consider an example of sketching the exponential function.

Example 9.3.2

Draw the graph for the given function, $\mathcal{F}$ (x) = $2^x - 1$.

Solution:

The domain of this function is all real numbers. As x goes to negative infinity, the function approaches zero, and as x goes to infinity, the function approaches infinity.

The base of the exponential function is 2, which is greater than one, so the function increases as x increases.

To find critical points, we need to find where the derivative is zero. f'(x) = $2^x\ln(2)$. This derivative is zero only when x = 0.

Thus, the critical point occurs at x = 0. To determine the concavity, we can find the second derivative of the function. f"(x) = $2^x \ln(2)$. Since the second derivative is always positive, the graph must be convex upward everywhere.

Now we just need to find the value of the function at the critical point. f(0) = 20 – 1 = 0.

We can now use this information to sketch the graph of the exponential function.

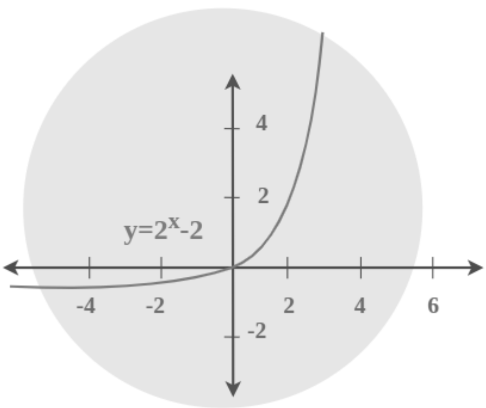


Figure 9.3.2 Graph of $\mathcal{F}$ (x) = 2^x – 1

Logarithmic Functions

We know that logarithmic functions are inverse of exponential functions. The function y = $\log_b x$ is the inverse of y = b^x. The graph of the exponential function is given below. We also know that the graph of an inverse of a function is basically a mirror image of the graph in y = x. So we can derive the shape of the graph of log function from the given graph of the exponential function.

The mirror image of the Logarithmic function is the exponential function both of them are shown in the image below,

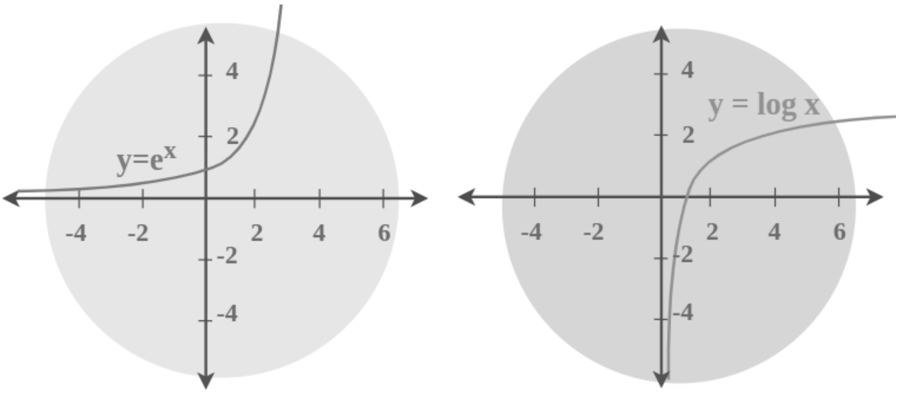


Figure 9.3.3 Graph of $\mathcal{F}$ (x) = logx

Let's see an example of graphic logarithmic functions.

Example 9.3.3

Plot the graph for $\log_{10}x + 5$.

Solution:

We can see that the function is f(x) = $\log_{10}x + 5$.

The graph of this equation will be shifted 5 units in the upwards direction.

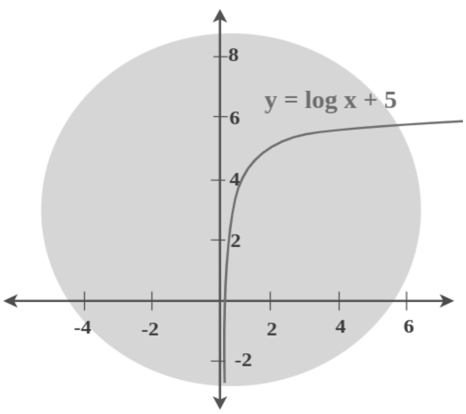


Figure 9.3.4 Graph of $\mathcal{F}$ (x) = $\log_{10}x + 5$

9.4 Summary

Tracing curves involves analyzing the behavior of functions and their graphs, including understanding key features such as intercepts, asymptotes, local extrema, concavity, and end behavior. Tracing curves typically involves plotting points, determining critical points and intervals, analyzing derivatives and second derivatives, identifying symmetries, and understanding transformations. Linear functions are functions that can be represented by a straight line when graphed on a Cartesian coordinate system. A linear function has the form f(x) = mx + b, where m represents the slope of the line and b represents the y-intercept. Polynomial functions are functions that consist of terms involving powers of a variable multiplied by coefficients. Polynomial functions can have multiple terms and can exhibit various behaviors such as oscillations, local maxima and minima, and end behavior determined by the highest degree term.

9.5 Keywords

- Tracing of curve
- Polynomial Functions
- Logarithmic Functions

9.6 Self – Assessment Questions

1. What is meant by "tracing a curve"?
2. What are some common methods used to trace curves?
3. What role do derivatives play in tracing curves?
4. How does the sign of the derivative help in understanding the behavior of a curve?
5. What information does the first derivative provide about a curve?
6. How do critical points help in tracing curves?
7. What does the second derivative reveal about the curve?
8. What is the significance of inflection points in tracing curves?
9. How do asymptotes affect the tracing of curves?
10. Can technology aid in tracing curves, and if so, how?

9.7 Case Study

1. If the curve depicts the pace at which a heated object cools over time, explain how the curve varies as the object gets closer to room temperature.

2. List the months with the highest and lowest sales figures. How did the curve help you determine these points?

9.8 References

1. Thomas, G. B., Weir, M. D., Hass, J., & Giordano, F. R. (2017). Thomas' Calculus: Early Transcendentals (14th ed.). Pearson.
2. Edwards, C. H., & Penney, D. E. (2015). Calculus: Early Transcendentals (7th ed.). Pearson.

CHAPTER -10

Concavity and Convexity

Learning Objectives:

- Recognize the geometric properties of concave and convex functions, including their shapes and curvature.
- Understand the criteria for determining points of inflection, such as changes in concavity.
- Apply concavity and convexity to solve optimization problems and analyze the behaviour of functions in various contexts.

Structure:

10.1 Introduction:

In the realm of Cartesian coordinates, understanding the concavity and convexity of curves provides essential insights into their overall shape and behaviour.

- **Concavity**: A curve is said to be concave if it curves downward like the interior of a bowl. This is characterized by a negative second derivative of the curve with respect to its independent variable.
- **Convexity**: Conversely, a curve is termed convex if it curves upward, resembling the outer surface of a bowl. This is indicated by a positive second derivative of the curve.

Convex functions appear in many problems in pure and applied mathematics. They play an extremely important role in the study of both linear and non linear programming problems. It is very important in the study of optimization. The solutions to these problems lie on their vertices.

The theory of convex functions is part of the general subject of convexity, since a convex function is one whose epigraph is a convex set. Nonetheless it is an important theory which touches almost all branches of mathematics. Graphical analysis is one of the first topics in mathematics which requires the concept of convexity. Calculus gives us a powerful tool in recognizing convexity, the second-derivative test. Miraculously, this has a natural

Curve Sketching as its name suggests helps us sketch the approximate graph of any given function which can further help us visualize the shape and behaviour of a function graphically. Curve sketching isn’t any sure-shot algorithm that after application spits out the graph of any desired function but it is an active role approach for a visual representation of a function that needs analysis of various features of graphs, such as intercepts, asymptotes, extreme, and concavity, to gain a better understanding of how the function behaves.

In this article, we will explore all the fundamentals of curve sketching and its solved examples. Other than that we will also explore all the aspects in detail which will help us analyze and sketch the function more efficiently.

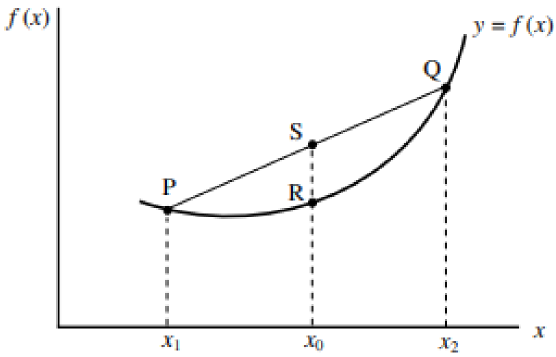


Figure 10.1.1.: Convex Function

Convexity of the curve is a reflection of a diminishing MRS (Marginal Rate of Substitution), which simply means that, as consumption of any one good increases more and more, the individual will prefer to sacrifice lesser and lesser amounts of consumption of the other good, that is $\frac{d^2y}{dx^2} \geq 0$. This implies that slope of the tangent to the curve declines as we move down the indifference curve. This implies that slope of the tangent to the curve declines as we move down the indifference curve

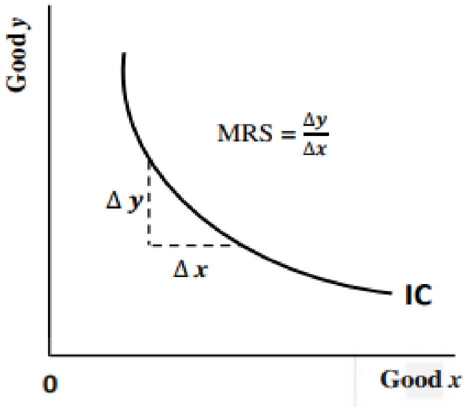


Figure 10.1.2 Slope of an Indifference Curve

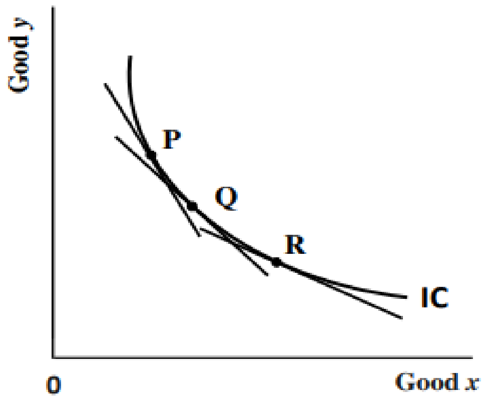


Figure 10.1.3 Diminishing MRS

10.2 Curve Sketching Definition

Curve Sketching is a collection of various techniques which can be used to create the approximate graph of any given function. That can help us analyze different features and behaviour of the graph. Curve Sketching involves analysis of many aspects of a given function such as changes in function as input changes, maximum and minimum values, intercepts, domain, range, asymptotes, etc. Curve Sketching is used to visualize and understand the shape and behaviour of any given function.

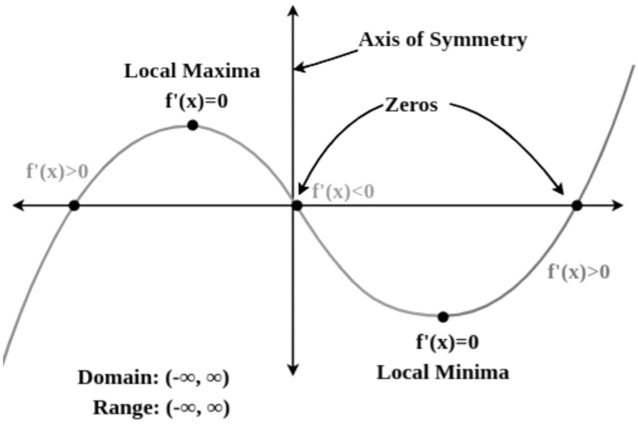


Figure 10.2.1 Curve Sketching

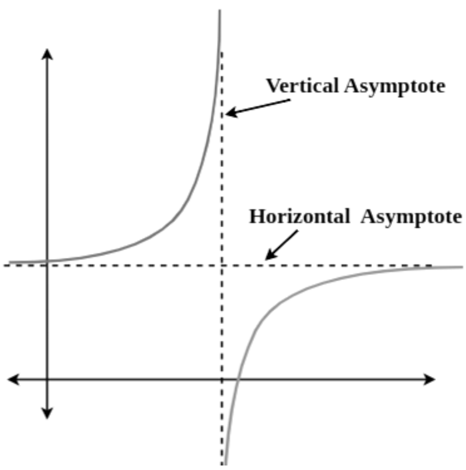


Figure 10.2.2 Curve Sketching

Graphing Basics

To create a graph of any given function, we need to plot some points such as intercepts, critical points, and some regular points which can help us trace the graph on the cartesian plane. Let's further understand these basics in detail as follows:

Plotting Points

We can easily plot various different points of any function on the graph by just using random input and their outputs as the coordinates. This random plot of points helps us connect the final graph after all the necessary calculations are done. For example, we need to graph the function $f(x) = e^x$, so just putting $x = \log_e 3$ we get the output $f(\log_e 3) = 3$. Now, we can $(\log_e 3, 3)$ as a point on the graph.

Domain and Range

First, analyze the function (f) to check for its domain and obtained points where the function's value becomes undefined or is discontinuous. For example: 1/x is not defined at x = 0. Log(x) is defined only at positive values of x.

The domain of a function f is the set of all values x for which f(x) is defined.

The range of a function f is the set of all values that f(x) takes on as a runs through the domain of f. That is, it is the set of all y values for which there is an a value such that

y = f(x).

For many functions, the domain is easy to determine. Often, all we have to do is look for which z cause a problem when evaluating f(x); those x are not in the domain of f, and the domain is everything else.

For example, suppose f(x)= 5/x(x-1). To compute f(x) we have to multiply x times x-1. This part causes no trouble: we can multiply any two number together. Then, we divide 5 by the product just calculated. This goes well, unless that product is zero. That is, the only z values that cause trouble are those for which

x(x-1)=0.

Since the left hand side of this equation is factored, we see that the only for which x(x-1) = 0 are x = 0 and 1. Thus, those are the only two values not in the domain of f, and so the domain of f is everything else. We might say: the domain of f is all $x \neq 0$, $x \neq 1$.

10.3 Summary

Concavity and convexity are concepts used to describe the shape of curves and surfaces. They are fundamental in calculus and mathematical analysis for understanding the behavior of functions and optimizing problems. Curve sketching is the process of visualizing and understanding the behavior of a function by examining its graph. The main objective of curve sketching is to identify key features of the graph such as intercepts, asymptotes, symmetry, intervals of increase/decrease, local extrema, concavity, and points of inflection.

10.4 Keywords

- Concavity
- Convexity
- Domain and Range

10.5 Self – Assessment Questions

1. Define concavity and convexity in terms of curves.
2. What is a concave-up (convex) curve?
3. What is a concave-down (concave) curve?
4. How can you determine concavity or convexity using the second derivative test?
5. What does it mean for a function to be concave or convex on an interval?
6. Can a function be both concave and convex on the same interval?
7. How do inflection points relate to concavity and convexity?
8. What is the significance of the sign of the second derivative in determining concavity or convexity?
9. How can you sketch a curve based on its concavity and convexity?

10.6 Case Study

1. Discuss how the concavity of the function helps in solving this optimization problem.
2. Consider a consumer deciding how much of two goods, x and y, to purchase with a fixed income. The consumer's utility function is given by $U(x, y)$, which is concave.

10.7 References

1. Anton, H., Bivens, I., & Davis, S. (2010). Calculus Early Transcendentals (9th ed.). John Wiley & Sons.

2. Larson, R., & Edwards, B. (2017). Calculus of a Single Variable (11th ed.). Cengage Learning.

CHAPTER -11

Asymptotes

Learning Objectives:

- Recognize the different types of asymptotes, including vertical, horizontal, and oblique asymptotes.
- Identify the equations and properties of vertical, horizontal, and oblique asymptotes.
- Apply knowledge of asymptotes to analyze the behaviour of functions near points of discontinuity or singularity.

Structure:

11.1 Finding Intercepts and Asymptotes

Intercepts are the points where the graph cuts the coordinate axis and to find the x-intercept, we put y = 0 and solve for x. Similarly, to find the y-intercept, we put x = 0 and solve for y.

Asymptotes are lines that the graph approaches but do not intersect. There are three types of asymptotes which are as follows:

- Horizontal Asymptote
- Vertical Asymptote
- Slant Asymptote

Horizontal asymptotes occur when the function approaches a constant value as x approaches infinity or negative infinity. Vertical asymptotes occur when the function approaches infinity or negative infinity as x approaches a specific value. Slant asymptotes occur when the degree of the numerator is one more than the degree of the denominator.

To calculate Horizontal Asymptote, we need to calculate the limit of a function at infinity, and vertical asymptotes are those points for which functions become not defined i.e., the denominator becomes 0.

Example 11.1.1

Find Intercept and Asymptote for f(x) = (2x + 1) / (x – 3).

Solution:

To find the x-intercept, we set f(x) = 0 and solve for x:

$\Rightarrow (2x + 1) / (x - 3) = 0$

$\Rightarrow 2x + 1 = 0 \ (x \neq 3)$

$\Rightarrow x = -1/2$

Therefore, the x-intercept of the function is at (-1/2, 0).

To find the y-intercept, we set x = 0 and solve for f(x):

$\Rightarrow f(0) = (2(0) + 1) / (0 - 3) = -1/3$

Therefore, the y-intercept of the function is at (0, -1/3).

The vertical asymptote occurs at x = 3, since the denominator of the function becomes zero at that point.

To find the horizontal asymptote, we need to examine the behaviour of the function as x approaches infinity or negative infinity. We can do this by dividing the numerator and denominator by the highest power of x in the function:

$f(x) = (2x + 1) / (x - 3) = (2 + 1/x) / (1 - 3/x)$

As x becomes very large or very small, the term 1/x becomes insignificant compared to the other terms in the numerator and denominator, so we can ignore it:

$f(x) \approx 2 / 1 = 2$ (as $x \to \pm\infty$)

Therefore, the horizontal asymptote of the function is $y = 2$.

11.2 Local Extrema and Inflection Points

Local extrema are those points of the function or graph for which there is no such value of function greater or smaller than the local extrema i.e., no other point in the neighbourhood of the local extrema has a more extreme value than it.

To find out the maxima and minima in any function, we need to find the critical points. Critical points of the function are defined as the points where either slope of the function is not defined or the slope is 0 i.e., $f'(x) = 0$.

After getting the values of critical points, check the second derivative of the function at those critical points. If $f''(x) > 0$ for some critical point $x=k$, then $f(k)$ is the local minima of the function, and if $f''(x) < 0$ for some critical point $x = k$, then $f(k)$ is the local maxima of the function.

If $f''(x) = 0$ for some critical point $x = k$ then $x = k$ is the Point of Inflection or Inflection Point of the function.

11.3 Calculating Slope and Concavity

The slope is the measure of inclination from the positive x-axis and it tells us whether the graph is increasing (Slope>0) or decreasing (Slope<0). To find the slope of any given function, we differentiate the given function w.r.t to the dependent variable and substitute the value for which we need to calculate the slope.

Concavity is the measure of the curve which tells us whether the graph is concave up or concave down i.e., the direction of curvature of the graph. To calculate the concavity, we use the second derivative w.r.t dependent variable of the function. If the second derivative is positive, then the function is concave up, and if the second derivative is negative, then the function is concave down.

Example 11.3.1

Find the slope and concavity of $f(x) = x^3 – 3x^2 + 2x$.

Solution:

$f(x) = x^3 – 3x^2 + 2x$

$\Rightarrow f'(x) = 3x^2 – 6x + 2$

To find the slope of the function at a specific point, we substitute the value of x in the derivative:

$f'(-1) = 3(-1)^2 – 6(-1) + 2 = 11$

Therefore, the slope of the function at x = -1 is 11.

Obtain the function's concavity,

$f''(x) = 6x – 6$

To find the points where the concavity changes, we set $f''(x) = 0$ and solve for x:

$\Rightarrow f''(x) = 6x – 6 = 0$

$\Rightarrow x = 1$

Therefore, the function changes from concave down to concave up at x = 1.

Here, (1, 0) is the inflection point.

Inflection Point: Inflection point is the point where the concavity changes i.e., the second derivative of function = 0.

11.4 Summary

An asymptote is a line or curve that a function approaches but never reaches as the independent variable approaches a certain value or infinity. Vertical asymptotes occur when the function approaches positive or negative infinity as the independent variable approaches a certain value. Horizontal asymptotes occur when the function approaches a constant value as the independent variable approaches positive or negative infinity. Oblique (or slant) asymptotes occur when the function approaches a straight line as the independent variable approaches positive or negative infinity. To find asymptotes, analyze the behavior of the function as approaches certain values or infinity. For vertical asymptotes, look for values of a that make the denominator of a rational function zero but not the numerator. For horizontal and oblique asymptotes, analyze the end behavior of the function. Symmetry refers to a property where one part of an object mirrors or reflects the other part.

11.5 Keywords

- Asymptotes
- Local Extrema
- Inflection Points

11.6 Self – Assessment Questions

1. What is an asymptote?
2. How can you identify vertical asymptotes?
3. What is the significance of a horizontal asymptote?
4. Can a curve have more than one horizontal asymptote?
5. How are oblique (slant) asymptotes different from horizontal and vertical asymptotes?
6. Can a curve intersect its asymptotes?
7. What are the conditions for a rational function to have an oblique asymptote?
8. Can exponential or logarithmic functions have asymptotes?
9. How do you find the equations of asymptotes?

11.7 Case Study

1. In factories, the quantity, x, of an item produced determines its cost of production. The formula C(x) = 125x + 2000 provides the cost for a specific factory. This shows that there is an initial setup cost of $2000 for the production floor and that each item costs $125. The cost function is divided by the quantity, x, to determine the average cost of producing x products. In this instance, the typical cost function is

$$f(x) = (125x + 2000)/x$$

   Many other applications require finding averages in a similar way. Written without a variable in the denominator, this function will contain a negative integer power.

2. A manufacturer needs to be paid $50,000 to set up a production line before a company can make the next big toy. In addition, they must pay labor and raw materials costs of $5 per item. For the average cost to create x things, write a function. After that, explain what happens to the average cost when the firm produces a big quantity of toys.

11.8 References

1. Stewart, J. (2016). Calculus: Early Transcendentals (8th ed.). Cengage Learning.
2. Larson, R., & Edwards, B. (2017). Calculus of a Single Variable (11th ed.). Cengage Learning.

CHAPTER -12

Parametric Equations

Learning Objectives:

- Understand the advantages of using parametric equations to represent curves and trajectories in various contexts.
- Recognize the graphical representation of parametric curves and their geometric properties.
- Apply knowledge of parametric equations to analyze the behaviour of curves represented parametrically.

Structure:

12.1 Introduction to Parametric Equations:

Parametric equations are a way of representing curves or surfaces using one or more independent parameters. Rather than expressing the relationship between variables directly, parametric equations describe how each coordinate of a point on the curve varies as a function of one or more parameters. The motivation behind parametric equations often arises from situations where describing a curve or surface solely in terms of one variable isn't sufficient. For example, consider the motion of a particle in space, where both the x, y, and z coordinates change over time. Parametric equations allow us to track each coordinate's motion independently by introducing parameters that represent time or other variables.

Examples of Curves Represented Parametrically:

1. The Unit Circle:

Parametric equations for the unit circle (a circle with radius 1 centered at the origin) are given by:

$x = \cos(t)$

$y = \sin(t)$

Here, t ranges from 0 to 2π. As t varies, the x and y coordinates trace out the points along the circumference of the circle. This representation provides a clear understanding of how the coordinates change as the angle t varies.

2. The Ellipse:

The parametric equations for an ellipse centered at the origin with semi-major axis a along the x-axis and semi-minor axis b along the y-axis are:

$x = a \cos(t)$

$y = b \sin(t)$

Here, t again ranges from 0 to 2π. These equations describe how the coordinates of points on the ellipse change as the parameter t varies. By adjusting the values of a and b, we can generate ellipses of different sizes and eccentricities.

These examples illustrate how parametric equations offer a powerful way to describe geometric shapes and trajectories by breaking them down into simpler, more manageable components. They allow us to study and manipulate complex curves and surfaces with greater precision and flexibility than traditional Cartesian equations.

12.2 Parameterizing a Curve

Parameterizing a curve essentially means representing it in terms of one or more parameters. This allows us to describe each point on the curve using the parameter(s), enabling us to analyze and manipulate the curve more easily. There are various techniques for parameterization, including using trigonometric functions, polynomials, or a combination of both.

1. Trigonometric Functions:

• Trigonometric functions like sine and cosine are commonly used for parameterization, especially for circular or periodic curves. For example, consider the unit circle $x^2 + y^2 = 1$. It can be parameterized as $x = \cos(t)$ and $y = \sin(t)$, where t ranges from 0 to 2π. This parameterization traces out the circle as t varies.

2. Polynomials:

Polynomials are used to parameterize curves with more complex shapes. For instance, a line segment between two points (x_1, y_1) and (x_2, y_2) can be parameterized as $y = x_1 + (x_2 - x_1)t$ and $y = y_1 + (y_2 - y_1)t$, where t $(y_2\text{-}y_1)t$, ranges from 0 to 1.

• Higher degree polynomials can be used for more intricate curves. For example, a parabola $y = ax^2 + bx + c$ can be parameterized by letting $a = t$ and $y = at^2 + bt + c$.

3. Combination of Trigonometric Functions and Polynomials:

Sometimes, a combination of trigonometric functions and polynomials is used for parameterization. For example, the path of a projectile can be parameterized using both linear and quadratic functions to account for both horizontal and vertical motions.

4. Examples:

Circular Helix: Parameterized as x = cos(t), y = sin(t), and z = t. This traces out a helix in 3D space.

Bezier Curves: Parameterized using Bernstein polynomials. For example, a quadratic Bezier curve between points P_0 and P_2 with control point P_1 can be parameterized as $B(t) = (1-t)^2 P_0 + 2t(1-t)P_1 + t^2 P_2$.

• Elliptic Curve: Parameterized as x = a a cos(t) and y = b sin(t), where a and b are constants determining the shape of the ellipse.

These examples demonstrate how different parameterizations can be used to describe various types of curves, from simple lines to complex geometric shapes. The choice of parameterization depends on the specific characteristics of the curve and the requirements of the problem at hand.

12.3 Length of Parametric Curves

The arc length formula for parametric curves allows us to calculate the length of a curve defined by parametric equations. Parametric curves are defined by two functions, typically denoted as (t) and (t), where t is the parameter that traces out the curve.

The arc length formula for a parametric curve is given by:

$$S = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Where:

S represents the arc length of the curve.

a and b are the limits of the parameter t over which the curve is traced.

dx/xt and dy/dt are the derivatives of x(t) and y(t) respectively with respect to t.

Derivation:

To understand the significance of this formula, let's briefly derive it.

Consider a small arc length ds on the curve defined by the parametric equations x(t) and y(t).

By the Pythagorean theorem, we have:

$ds^2 = dx^2 + dy^2$

Where dx and dy are infinitesimal changes in x and y respectively.

Dividing both sides by dt^2 and taking the square root, we get:

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

Integrating both sides over the interval [a, b], we obtain the arc length formula.

Applications:

1. **Engineering and Design**: In architecture, civil engineering, and design, understanding the length of curves is crucial for tasks such as designing roadways, pipelines, and bridges.
2. **Physics and Mechanics**: In physics, the trajectory of a particle in space can be described parametrically, and calculating its arc length helps in understanding its motion.
3. **Computer Graphics and Animation**: In computer graphics, parametric curves are used to define paths for animations and simulations. Calculating arc lengths helps in creating realistic motion.
4. **Robotics and Motion Planning**: In robotics, understanding the length of curves is essential for motion planning, obstacle avoidance, and path optimization.
5. **Biology and Medicine**: In biological studies, parametric curves can represent biological structures such as blood vessels or organ surfaces, and calculating arc lengths aids in analyzing their shapes and properties.

In essence, the arc length formula for parametric curves is a powerful tool with wide-ranging applications in various fields where understanding the length of curves is essential.

12.4 Summary

1. Definition:

- Parametric equations are a set of equations that express the coordinates of a point as functions of one or more independent variables, called parameters.
- Each coordinate (such as x and y in two dimensions or x, y, and z in three dimensions) is given as a function of the parameter(s).
- Parametric equations are often used to describe curves, trajectories, and motion in both mathematics and physics.

2. Form:

In two dimensions, parametric equations are typically written as:

$x = f(t)$

$y = g(t)$

In three dimensions, parametric equations can be written as:

x = f(t)

y = g(t)

z = h(t)

Here, t is the parameter, and f(t), g(t), and h(t) are functions that describe the motion or trajectory of the point.

3. Advantages:

- Parametric equations are particularly useful for describing complex curves and motion that cannot be easily expressed by a single function y = f(x).
- They allow for a more flexible representation of curves, including curves with multiple branches, loops, and cusps.

12.5 Keywords

- Parametric Equations
- Parameterizing a Curve

12.6 Self – Assessment Questions

1. What are parametric equations?
2. How do parametric equations represent curves?
3. What is the purpose of introducing parameters in equations?
4. Can parametric equations describe curves that cannot be expressed by Cartesian equations?
5. How do you eliminate the parameter to express a curve in Cartesian form?
6. What is the significance of the parameter in parametric equations?
7. How are parametric equations used to describe motion in physics?
8. Can parametric equations represent curves in higher dimensions?
9. What are the advantages of using parametric equations over Cartesian equations in certain situations?
10. How are parametric equations graphed or plotted?

12.7 Case Study

1. Find the position, point (x, y), of a projectile

   A golf ball is hit on level ground with an initial velocity of 55 m/s at an angle of 40∘ with respect to the horizontal. Where will the ball be 2 seconds later?

2. Find the maximum height of a projectile.

   A golf ball is hit on level ground with an initial velocity of 55 m/s at an angle of 40° with respect to the horizontal. What is the maximum height of the golf ball?

12.8 References

1. Stewart, J. (2016). Calculus: Early Transcendentals (8th ed.). Cengage Learning.
2. Larson, R., & Edwards, B. (2017). Calculus of a Single Variable (11th ed.). Cengage Learning.

CHAPTER -13

Multiple Integrals and Change of Order of Integration

Learning Objectives:

- Define what multiple integrals are and their significance in calculus and mathematical analysis.
- Learn techniques for evaluating multiple integrals, including iterated integration and using appropriate coordinate systems (rectangular, polar, cylindrical, spherical).
- Understand the concept of changing the order of integration in double and triple integrals.

Structure:

13.1 Introduction to Multiple Integrals:

Integration is the inverse process of differentiation. In the differential calculus, we are given a function and we have to find the derivative or differential of this function, but in the integral calculus, we are to find a function whose differential is given. Thus, integration is a process which is the inverse of differentiation. Let $\frac{d}{dx}F(x) = f(x)$. Then we write $\int f(x)dx = F(x) + C$. These integrals are called indefinite integrals or general integrals; C is called constant of integration. All these integrals differ by a constant.

From the geometric point of view, an indefinite integral is collection of family of curves, each of which is obtained by translating one of the curves parallel to itself upwards or downwards along the y-axis.

Some properties of indefinite integrals are as follows:

1. $\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$

2. $\int k\ f(x)\,dx = k \int f(x)dx$

More generally, if $f_1, f_2, \dots\dots, f_n$ are functions and $k_1, k_2, \dots\dots, k_n$ are real numbers. Then

$$\int\left[k_1 f_1(x) + k_2 f_2(x) + \dots\dots + k_n f_n(x)\right]dx = k_1\int f_1(x)\,dx + k_2\int k_2(x)\,dx + \dots + k_n\int f_n(x)\,dx$$

Some standard integrals:

(i) $\int x^n dx = \dfrac{x^{n+1}}{n+1} + C, n \neq -1.$ Particularly, $\int dx = x + C$

(ii) $\int \cos x \, dx = \sin x + C$

(iii) $\int \sin x \, dx = -\cos x + C$

(iv) $\int \sec^2 x \, dx = \tan x + C$

(v) $\int \operatorname{cosec}^2 x \, dx = -\cot x + C$

(vi) $\int \sec x \tan x \, dx = \sec x + C$

(vii) $\int \operatorname{cosec} x \cot x \, dx = -\operatorname{cosec} x + C$

(viii) $\int \dfrac{dx}{\sqrt{1-x^2}} = \sin^{-i} x + C$

(ix) $\int \dfrac{dx}{\sqrt{1-x^2}} = -\cos^{-i} x + C$

(x) $\int \dfrac{dx}{1+x^2} = tan^{-1}x + C$

(xi) $\int \dfrac{dx}{1+x^2} = -cot^{-1}x + C$

(xii) $\int e^x dx = e^x + C$

(xiii) $\int a^x dx = \dfrac{a^x}{\log a} + C$

(xiv) $\int \dfrac{dx}{x\sqrt{x^2-1}} = \sec^{-1} + C$

(xv) $\int \dfrac{dx}{x\sqrt{x^2-1}} = -\operatorname{cosec}^{-1} + C$

(xvi) $\int \dfrac{1}{x} dx = \log|x| + C$

Integration by Partial Fraction: A rational fraction is the ratio of the form $\frac{P(x)}{Q(x)}$ where P(x) and Q(x) are polynomials in x and Q(x) $\neq$ 0. If the degree of the polynomial P(x) is greater than the degree of the polynomial Q(x), then we may divide P(x) by Q(x) so that $\frac{P(x)}{Q(x)} = T(x) + \frac{P_1(x)}{Q(x)}$ where T(x) is a polynomial in x and degree of $P_1(x)$ is less than Q(x). T(x) is a polynomial can be easily integrated.

$\frac{P_1(x)}{Q(x)}$can be integrated by expressing $\frac{P_1(x)}{Q(x)}$ as the sum of partial fractions of the following types:

(a) $\frac{px+q}{(x-a)(x-b)} = \frac{\mathrm{A}}{x-a} + \frac{\mathrm{B}}{x-b} \quad x \neq a, x \neq b, a \neq b$

(b) $\frac{px+q}{(x-a)^2} = \frac{\mathrm{A}}{x-a} + \frac{\mathrm{B}}{(x-a)^2}$

(c) $\frac{px^2+qx+r}{(x-a)(x-b)(x-c)} = \frac{\mathrm{A}}{x-a} + \frac{\mathrm{B}}{x-b} + \frac{\mathrm{C}}{x-c}$

(d) $\frac{px^2+qx+r}{(x-a)^2(x-b)} = \frac{\mathrm{A}}{x-a} + \frac{\mathrm{B}}{(x-a)^2} + \frac{\mathrm{C}}{x-b}$

(e) $\frac{px^2+qx+r}{(x-a)(x^2+bx+c)} = \frac{\mathrm{A}}{x-a} + \frac{\mathrm{B}x+\mathrm{C}}{x^2+bx+c}$

where x^2+bx+c cannot be factorize further into linear fraction.

Integration by Substitution: In this method we change the variable to some other variable. When the integrand involves some trigonometric functions, we shall be using some well-known identities to find the integrals. Using substitution technique, we obtain the following standard integrals:

(a) $\int \tan x dx = \log|\sec x| + c$

(b) $\int \cot x dx = \log|\sin x| + c$

(c) $\int \sec x dx = \log|\sec x + \tan x| + c$

(d) $\int \cos ecx dx = \log|\cos ecx - \cot x| + c$

Integrals of Some Special Functions:

(a) $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| + c$

(b) $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \log \left| \frac{a+x}{a-x} \right| + c$

(c) $\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} + c$

(d) $\int \frac{dx}{\sqrt{x^2 - a^2}} = \log \left| x + \sqrt{x^2 - a^2} \right| + c$

(e) $\int \frac{dx}{\sqrt{a^2 + x^2}} = \log \left| x + \sqrt{a^2 + x^2} \right| + c$

(f) $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a} + c$

Integration by Parts: For the given function f(x) and g(x),

$$\int f(x) . g(x)\, dx = f(x) \int g(x)\, dx - \int \left\{ f'(x) \int g(x)\, dx \right\} dx$$

We must take proper care to choose the first function and second function clearly. We must choose that function as the second function whose integral is well-known to us.

- $\int e^x \left[f(x) + f'(x) \right] dx = e^x f(x) + c$

Some Special Type of Integrals:

(a) $\int \sqrt{a^2 - x^2} dx = \frac{1}{2} \left[x\sqrt{a^2 - x^2} + a^2 \sin \frac{x}{a} \right]$

(b) $\int \sqrt{x^2 + a^2} dx = \frac{1}{2} \left[x\sqrt{x^2 + a^2} + a^2 \log \left| x + \sqrt{x^2 + a^2} \right| \right] + c$

(c) $\int \sqrt{x^2 - a^2} dx = \frac{1}{2} \left[x\sqrt{x^2 - a^2} + a^2 \log \left| x + \sqrt{x^2 - a^2} \right| \right] + c$

Integrals of the types $\int \frac{dx}{ax^2 + bx + c}$ or $\int \frac{dx}{\sqrt{ax^2 + bx + c}}$

(a) These type of integrals are transformed into standard form by expressing

$$ax^2 + bx + c = a \left(x^2 + \frac{bx}{a} + \frac{c}{a} \right) = a \left[\left(x + \frac{b}{2a} \right)^2 + \left(\frac{c}{a} - \frac{b^2}{4a^2} \right) \right]$$

(b) Integrals of the types $\int \frac{px+q}{ax^2+bx+c}dx$ or $\int \frac{px+q}{\sqrt{ax^2+bx+c}}dx$ are transformed into standard form by expressing $px+d = \mathrm{A}\frac{d}{dx}\left(ax^2+bx+c\right)+\mathrm{B}$

$\Rightarrow \quad px+d = \mathrm{A}\left(2ax+b\right)+\mathrm{B}$

where A and B are determined by comparing coefficients on both sides.

- We have already defined $\int_a^b f(x)\,dx$ as the area of the region bounded by the curve $y = f(x)$, $a \le x \le b$, the x-axis and the ordinates x = a and x = b. Let x be a given point in [a, b], then $\int_a^b f(x)\,dx$ represents the area function A(x).
- First Fundamental Theorem of Integral Calculus: Let the area function be defined by $\mathrm{A}(x) = \int_a^b f(x)\,dx$ for all $x \ge a$ where f be assumed to be continuous on [a, b], then $\mathrm{A}'(x) = f(x)$ for all $x \in [a, b]$.
- Second Fundamental Theorem of Integral Calculus: Let f be a continuous function of x defined on the closed interval]a, b] and let F be another function such that $\frac{d}{dx}\mathrm{F}(x) = f(x)$ for all x in the domain of f, then $\int_a^b f(x)\,dx = \mathrm{F}(x) + \left.\mathrm{C}\right]_a^b = \mathrm{F}(b) - \mathrm{F}(a)$

This is called the definite integral of f over the range [a, b] were a and b are called the limits of integration, a being the lower limit and b the upper limit.

Multiple integrals extend the concept of integration from one dimension to multiple dimensions. While single integrals deal with functions of one variable over a one-dimensional interval, multiple integrals handle functions of multiple variables over regions in two or more dimensions.

Definition and Concept of Multiple Integrals: A multiple integral is an extension of a single integral to functions of more than one variable. In essence, it represents the process of finding the signed volume (for double integrals) or the signed volume or hypervolume (for triple integrals) of a region in space bounded by a surface or surfaces.

For a double integral over a region R in the xy-plane of a function f(x, y), it's represented as:

$$\iint_R f(x, y)\, dA$$

Here, dA represents the infinitesimal area element in the xy-plane. Similarly, for a triple integral over a region E in three-dimensional space of a function f(x, y, z), it's represented as:

$$\iiint_E f(x, y, z)\, dV$$

Where *dV* represents the infinitesimal volume element in three-dimensional space.

Comparison with Single Integrals:

Single integrals compute the area under a curve in one dimension, while multiple integrals compute the volume under a surface or hyper volume under a hyper surface in two or more dimensions. Just as single integrals have definite and indefinite forms, multiple integrals can also be definite or indefinite.

In essence, single integrals are like slicing a shape into infinitely thin slices along one direction and summing up the areas of these slices, whereas multiple integrals involve slicing a shape into infinitesimally small pieces in multiple directions and summing up the volumes of these pieces.

Examples of double and triple integrals

Double Integrals:

1. Area under a Curve:

Consider the region bounded by the curves $y = x^2$ and $y = 4 - x^2$, for $0 \leq x \leq 1$. The area of this region can be calculated using a double integral:

$$\iint_R 1\, dA$$

where R represents the region bounded by the curves.

2. Volume Between Surfaces:

Given two functions z = f(x, y) and z = g(x, y) over a region R in the xy-plane, the volume between the surfaces and under the region can be computed as:

$$\iint_R |f(x, y) - g(x, y)|\, dA$$

Triple Integrals:

1. Volume of a Solid Region:

Let's say we have a solid region bounded below by the plane z = 0, above by the plane z = 4 - x - 2y, and on the sides by the planes x = 0, y = 0, and a + 2y= 2. The volume of this solid can be calculated using a triple integral:

$$\iiint_V 1\,dV$$

where V represents the solid region.

2. Mass of a Three-Dimensional Object:

If we have a three-dimensional object with variable density *p(x, y, z)*, the total mass *M* of the object can be calculated using:

$$M = \iiint_V \rho(x, y, z)\,dV$$

where *V* represents the volume of the object.

These are just a few examples, but double and triple integrals have a wide range of applications in physics, engineering, economics, and various other fields for computing various properties of multivariable functions and regions in space.

13.2 Change of Order of Integration

Changing the order of integration is a crucial technique in multivariable calculus that allows for simplification and more efficient computation of double and triple integrals. The motivation behind changing the order of integration primarily stems from the desire to convert the given integral into a form that is easier to evaluate. This can involve converting from an integral with respect to one variable first to one with respect to another variable first, or from an iterated integral in one order to another order.

Techniques for changing the order of integration in double and triple integrals

In double and triple integrals, there are several techniques for changing the order of integration:

1. **Fubini's Theorem**: This theorem states that if the function being integrated is continuous on a rectangle (or a rectangular solid in the case of triple integrals), then the order of integration can be changed without affecting the value of the integral.

2. **Geometric Considerations**: Understanding the geometric interpretation of the integral and the region of integration can guide the choice of the order of integration. For example, if the region of integration is more naturally described in terms of one variable before another, it may be beneficial to change the order accordingly.
3. **Symmetry**: Exploiting symmetry can often simplify the integration process. Sometimes, changing the order of integration can make symmetry more apparent and lead to easier evaluations.
4. **Polar, Cylindrical, or Spherical Coordinates**: For integrals involving circular, cylindrical, or spherical symmetry, it is often advantageous to switch to polar, cylindrical, or spherical coordinates, respectively. This change of variables can simplify the integral and make changing the order of integration more straightforward.

Geometric interpretation of changing the order of integration

The geometric interpretation of changing the order of integration involves visualizing how the region of integration is projected onto different coordinate planes or dimensions. By changing the order of integration, you essentially rearrange the process of slicing and summing over the region, which can lead to a more manageable integral. Geometrically, this corresponds to reorienting the axes or changing the direction of integration to better align with the shape and symmetry of the region being integrated over.

13.3 Applications of Multiple Integrals

Multiple integrals have various applications in mathematics and physics, as you've listed, but they are also widely used in engineering, economics, and even biology.

Finding volume, mass, and center of mass using multiple integrals, Calculation of moments of inertia

Here are some additional applications:

1. **Engineering Design and Analysis**: Engineers often use multiple integrals to calculate various properties of objects and systems. For example, they might use double integrals to find the area of irregularly shaped regions or triple integrals to determine the volume of complex three-dimensional structures. In structural engineering, multiple integrals are used to analyze stress distributions in materials.
2. **Electromagnetism**: In physics, multiple integrals are used extensively in the study of electromagnetism. For instance, in calculating electric and magnetic fields generated

by charged or current-carrying objects, engineers and physicists rely on multiple integrals over space to model and predict these fields accurately.

3. **Fluid Dynamics**: Multiple integrals are crucial in fluid dynamics for determining properties such as fluid flow rate, pressure distribution, and fluid forces acting on solid surfaces. Engineers use these principles in designing aerodynamic shapes for aircraft, optimizing hydraulic systems, and understanding natural phenomena like ocean currents.
4. **Economics and Finance**: In economics, multiple integrals are used in analyzing consumer and producer surplus, calculating total revenue and profit functions, and determining equilibrium prices and quantities in complex market models. In finance, they are employed in option pricing models and risk management techniques.
5. **Geography and GIS**: Geographic Information Systems (GIS) utilize multiple integrals for spatial analysis, such as calculating the area covered by a particular land-use type, estimating population density within specific regions, or determining optimal routes for transportation networks.
6. **Quantum Mechanics**: In quantum mechanics, multiple integrals appear in the calculation of probabilities of various outcomes of particle interactions, the determination of energy eigenvalues and Eigenfunctions of quantum systems, and the calculation of expectation values of physical observables.
7. **Biology and Medicine**: Multiple integrals are used in modeling biological systems, such as the distribution of substances within tissues, the flow of fluids within blood vessels, and the diffusion of chemicals in cellular environments. In medical imaging, they play a crucial role in reconstructing three-dimensional images from two-dimensional scans.
8. **Computer Graphics and Animation**: In computer graphics, multiple integrals are used for rendering three-dimensional scenes, calculating lighting and shading effects, and simulating realistic physical phenomena like fluid dynamics and cloth motion.

These are just a few examples of the broad range of applications of multiple integrals across various fields. They provide powerful mathematical tools for modeling and solving complex real-world problems.

13.4 Summary

1. Multiple Integrals:

Definition: Multiple integrals extend the concept of integration to functions of multiple variables. Instead of integrating over a single interval, region, or volume, multiple integrals involve integrating over higher-dimensional regions, such as surfaces or volumes in two or three dimensions.

• Types:

• Double Integrals: Integrals over a region in the plane, where the integrand is a function of two variables.

Triple Integrals: Integrals over a region in three-dimensional space, where the integrand is a function of three variables.

• Higher-order Integrals: Integrals over regions in higher-dimensional spaces, extending the concept to functions of four or more variables.

Applications: Multiple integrals are used extensively in physics, engineering, economics, and other fields to calculate quantities such as mass, volume, density, moment of inertia, probability, and more.

2. Change of Order of Integration:

Definition: Change of order of integration refers to the process of rearranging the order in which integrals are evaluated when performing multiple integrals.

• Benefits: Changing the order of integration can simplify the evaluation of multiple integrals, making them easier to compute or leading to alternative representations.

Conditions: Changing the order of integration is valid under certain conditions, such as when the region of integration is simple or when the integrand is well-behaved.

• Techniques:

For double integrals, changing the order often involves converting from Cartesian coordinates to polar coordinates or vice versa.

For triple integrals, changing the order may involve rearranging the order of integration with respect to the three variables or using cylindrical or spherical coordinates.

13.5 Keywords

- Integration

- Integration by Parts
- Multiple Integrals
- Change of Order of Integration

13.6 Self – Assessment Questions

1. What is a multiple integral?
2. How is a double integral different from a single integral?
3. What is the interpretation of a triple integral?
4. When do we use iterated integrals for multiple integrals?
5. What does the order of integration refer to?
6. Under what conditions can you change the order of integration?
7. What is Fubini's theorem?
8. How does changing the order of integration affect the limits of integration?
9. Can you interchange the limits of integration in multiple integrals?
10. What are some common applications of multiple integrals and changing the order of integration in real-world problems?

13.7 Case Study

1. Assume that the charge density ρ(x,y) in a two-dimensional region is changeable. This is known as the electric charge distribution. The entire region must be integrated over ρ(x,y) in order to determine the overall charge within the region. Explain the double integral setup for this calculation, as well as any situations in which reversing the integration order could make the work easier.
2. Heat Distribution in a Plate: Let us consider a thin metal plate that has an uneven distribution of heat. T(x,y) gives the temperature of the plate at any given point (x,y). Integrating T(x,y) throughout the whole plate yields the total heat content. In order to determine the total heat content, explain how the double integral would be built up and talk about the possible advantages of varying the sequence of integration.

13.8 References

1. Stewart, J. (2016). Calculus: Early Transcendentals (8th ed.). Cengage Learning.
2. Larson, R., & Edwards, B. (2017). Calculus of a Single Variable (11th ed.). Cengage Learning.

CHAPTER -14

Beta and Gamma Functions

Learning Objectives:

- Define what Beta and Gamma functions are and their significance in mathematics, particularly in calculus, analysis, and probability.
- Understand how to derive and apply various integral representations and series expansions for Beta and Gamma functions.
- Apply Beta and Gamma functions to solve problems in calculus, probability, statistics, and physics.

Structure:

14.1 Introduction to Beta and Gamma Functions

Definition and properties of the gamma function**:**

The gamma function, denoted by Γ(x), is defined for all complex numbers except non-positive integers. It extends the factorial function to non-integer arguments.

Its properties include:

- Γ(x) = (x - 1)!
- Γ(x) = ∫[0, ∞] t^(x - 1) * e^(-t) dt

Definition and properties of the beta function:

The beta function, denoted by B(p, q), is defined for positive real numbers p and q. It is closely related to the gamma function and is expressed as:

- B(p, q) = ∫[0, 1] t^(p - 1) * (1 - t)^(q - 1) dt
- B(p, q) = Γ(p) * Γ(q) / Γ(p + q)

Relationship between the gamma and beta functions:

The relationship between the gamma and beta functions is fundamental in mathematical analysis and arises from their definitions and properties. The beta function, denoted as B(p,q), is defined in terms of the gamma function $\Gamma(x)$, and this interconnection provides a deeper understanding of both functions.

1. Beta Function in Terms of Gamma Function:

The beta function is expressed in terms of the gamma function as follows:

$$B(p,q) = \frac{\Gamma(p)\cdot\Gamma(q)}{\Gamma(p+q)}$$

2. Reciprocal Relation:

There exists a reciprocal relation between the gamma function and the beta function:

$$B(p,q) = \frac{\Gamma(p)\cdot\Gamma(q)}{\Gamma(p+q)} = \frac{1}{q-1}\int_0^1 t^{p-1}(1-t)^{q-1}dt$$
$$\frac{1}{B(p,q)} = \frac{\Gamma(p+q)}{\Gamma(p)\cdot\Gamma(q)} = \int_0^\infty e^{-x}x^{p-1}dx$$

3. Symmetry Property:

The beta function exhibits a symmetry property:

$B(p,q) = B(q,p)$

This property implies that the beta function is symmetric with respect to its parameters p and q.

4. Relation to Integral Representation:

The beta function can be represented as an integral involving the gamma function:

$$B(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}dt$$

This representation highlights the integral nature of the beta function and its connection to the gamma function.

5. Product Representation:

The beta function can be represented as a product of two gamma functions:

$$B(p,q) = \frac{\Gamma(p)\cdot\Gamma(q)}{\Gamma(p+q)}$$

This product representation emphasizes the relationship between the beta and gamma functions and facilitates various mathematical manipulations.

14.2 Evaluating Integrals with Beta and Gamma Functions

Integrals involving gamma and beta functions often arise in various areas of mathematics and physics, particularly in the context of probability theory, statistics, and mathematical physics. These functions have properties that make them particularly useful for evaluating certain types of integrals. Here's a brief overview of the gamma and beta functions and some techniques for using them to evaluate integrals:

1. Gamma Function (Γ(x)): The gamma function is defined for all complex numbers except for the non-positive integers. It is denoted by Γ(x) and is defined as:

Γ(x) = ∫[0 to ∞] t^(x-1) * e^(-t) dt

Some key properties of the gamma function include:

- Γ(x) = (x - 1)!
- Γ(x + 1) = xΓ(x)
- Γ(1/2) = √π

2. Beta Function (B(x, y)): The beta function is also defined for positive real numbers x and y. It is denoted by B(x, y) and is defined as:

B(x, y) = ∫[0 to 1] t^(x-1) * (1-t)^(y-1) dt

Some properties of the beta function include:

- B(x, y) = Γ(x)Γ(y) / Γ(x + y)
- B(x, y) = B(y, x)

Techniques for Evaluating Integrals using Gamma and Beta Functions:

1. **Substitution**: Sometimes, a clever substitution involving gamma or beta functions can transform an integral into a form where these functions can be directly applied. For example, using substitution to rewrite the integral in terms of the gamma or beta function.
2. **Integration by Parts**: Integration by parts can be used in combination with properties of gamma and beta functions to simplify integrals. This technique often involves choosing appropriate functions to differentiate and integrate.
3. **Using Known Results**: Sometimes, it's possible to express the integrand in terms of known functions, and then use properties of those functions to evaluate the integral. For instance, expressing the integrand in terms of trigonometric or exponential functions.
4. **Special Functions**: Gamma and beta functions are special functions that have well-defined properties, which can be exploited to simplify integrals. For example, using properties like the duplication formula or Euler's reflection formula.

Applications to Various Problems:

1. **Probability Distributions**: Many probability distributions, such as the gamma distribution, beta distribution, and chi-square distribution, involve gamma and beta functions in their probability density functions and cumulative distribution functions.
2. **Mathematical Physics**: Integrals involving gamma and beta functions often arise in problems related to quantum mechanics, statistical mechanics, and electromagnetic theory.
3. **Engineering Applications**: These functions find applications in various engineering fields, including signal processing, control theory, and communication theory.
4. **Number Theory**: Gamma and beta functions also have connections to number theory, particularly through their relation to combinatorial coefficients and special sequences.

Application in Solving Differential Equations:

Beta and gamma functions often appear in the solutions of differential equations, particularly in problems involving integral transforms, such as Laplace transforms and Fourier transforms. For instance, when solving differential equations involving linear operators, gamma and beta functions can serve as building blocks for finding solutions through integral representations.

14.3 Summary

Both the Beta and Gamma functions have widespread applications in various branches of mathematics, including calculus, probability theory, statistics, and mathematical physics. They are foundational tools for solving complex problems involving integration, combinatorics, and continuous distributions.

14.4 Keywords

- Beta and Gamma Functions
- Differential Equations

14.5 Self – Assessment Questions

1. What is the Beta function?
2. What is the Gamma function?
3. How are the Beta and Gamma functions related?

4. What is the definition of the Beta function in terms of an integral?

5. How is the Gamma function defined for positive real numbers?

6. What are the properties of the Beta function?

7. What are the properties of the Gamma function?

8. Can you express the factorial function using the Gamma function?

9. How are the Beta and Gamma functions used in calculus and mathematical analysis?

10. What are some applications of the Beta and Gamma functions in various fields of science and engineering?

14.6 Case Study

Goods with a chance x of defects are produced at a production plant. The plant manager anticipates that this likelihood will be 4% even if she is unsure about x based on prior experience. She also adds a standard deviation of 2% to her estimate of 4% in order to quantify her uncertainty regarding x. The manager, after conferring with a statistician, determines to model her uncertainty about x using a Beta distribution. In order to match her priors regarding the anticipated value and the standard deviation of x, how should she adjust the two distribution parameters?

14.7 References

1. Abramowitz, M., &Stegun, I. A. (1972). Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables. Dover Publications.
2. Arfken, G. B., Weber, H. J., & Harris, F. E. (2013). Mathematical Methods for Physicists: A Comprehensive Guide (7th ed.). Academic Press.